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Abstract. This article is concerned with the relativistic Vlasov equation, for collisionless axisymmetric plasmas immersed in a strong magnetic field, like in tokamaks. It provides a consistent kinetic treatment of the microscopic particle phase-space dynamics. It shows that the turbulent transport can be completely described through WKB expansions.

Keywords. Turbulence theory, WKB method, plasma physics, Vlasov equation, nearly integrable systems, ordinary differential equations, dynamical system, averaging methods.

MSC: Primary 35Q83; Secondary 34E13, 34E20, 34C29, 34C46, 70H06, 70H08, 76F20.

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1. Introduction.

Understanding the transport processes in axisymmetric magnetically confined plasmas like in tokamaks is a challenging goal. Of special interest are the collective events in the absence of collisions and with a long-term view of several microseconds. The physical context is here turbulence theory, and that refers to the deterministic study during long-times of the Vlasov equations submitted to a strong Lorentz force.

The Vlasov equation is a transport equation for the time evolution of a distribution function $f(\cdot)$. The value $f(\tau,x,v)$ gives the number of charged particles per unit volume having at the time $\tau\in\mathbb{R}_+$ near the position $x\in\mathbb{R}^3$ the velocity $v\in\mathbb{R}^3$. Under the influence of a large exterior magnetic field $\mathbf{B}(\cdot)$, the expression $f(\cdot)$ is subjected to:

$$\partial_\tau f+\varepsilon^{-1}\, v\cdot\nabla_x f\pm\varepsilon^{-2}\,\left[v\times\mathbf{B}(x)\right]\cdot\nabla_p f\pm(\mathbf{E}+v\times\mathbf{B})\cdot\nabla_p f=0\,. \tag{1.1}$$

The origin of the dimensionless formulation (1.1) is explained in Subsection 3.1.1. The plus or minus sign ($\pm$) corresponds to the selection of positive or negative particles, of ions or electrons. From now on, we will work with the sign $-$. Then, the observation time can be adjusted in such a way that $\tau=1$ represents 10^{-4} seconds, which is before the sawtooth crash but after turbulence has developed. The parameter ε is small , with $\varepsilon:=10^{-4}\ll 1$. The function $(\mathbf{E},\mathbf{B})(\tau,x)$ is assumed to be known. It is introduced to reflect the influence of a self-consistent electromagnetic field. The velocity v is limited by the speed of light, 1 in dimensionless variables. It is linked to the relativistic impulsion p through the relations:

$$v(p)\,=\,p\,\left(1+|p|^2\right)^{-1/2}<1\,,\qquad p(v)\,=\,v\,\left(1-|v|^2\right)^{-1/2}\,. \tag{1.2}$$

The ordinary differential equation underlying (1.1) is:

$$\left\{\begin{array}{lll}\partial_\tau x_\varepsilon\,=\,\varepsilon^{-1}\,v_\varepsilon\,, & & x_\varepsilon(0)=x_0\,,\\ \partial_\tau v_\varepsilon\,=\,\varepsilon^{-2}\,\mathrm{c}(|v_\varepsilon|)\,\mathbf{B}(x)\times v_\varepsilon+\mathrm{N}(v_\varepsilon,\mathbf{E},\mathbf{B})\,, & & v_\varepsilon(0)=v_0\,,\end{array}\right. \tag{1.3}$$

with functions $\mathrm{c}(\cdot)$ and $\mathrm{N}(\cdot)$ adjusted according to:

$$0<\mathrm{c}(|v|):=(1-|v|^2)^{1/2}\,,\qquad \mathrm{N}(v,\mathbf{E},\mathbf{B}):=(1-|v|^2)^{1/2}\left((\mathbf{E}\cdot v)\,v-\mathbf{E}+\mathbf{B}\times v\right). \tag{1.4}$$

The analysis of (1.3) has a long history and many different facets. The basic motions underlying (1.3) are discussed in most plasma physics books [1, 22, 24]. Mathematically, several approaches are possible, the two most important being KAM theory [2, 6, 20] and gyrokinetics [4, 9, 18]. In continuation of works on rotating fluids [12, 15], an alternative method has been proposed in [13]. It allows to extend classical results in longer times. The discussion depends heavily on the properties of the inhomogeneous magnetic field $\mathbf{B}(\cdot)$. The dipole model has been covered in [13], in connection with the description of magnetospheres. Axisymmetric toroidal plasmas are considered here, with a view towards fusion devices. The difference between the two situations comes mainly from the reduced hamiltonian dynamics involving a potential well in the first case and some integrable system of pendulum type in the second case. While the general strategy is the same, these two distinct geometrical frameworks require specific studies. In comparison with [13], major adaptations are needed. They make all the interest of the present contribution.

The description of trapped-electron mode turbulence is complex. It seems unpredictable because it is sensitive to small differences in positions or velocities. The motion is often viewed as chaotic. However, a complete analysis is available. Under the influence of $\mathbf{B}(\cdot)$, the charged particles act synchronously by organizing themselves as oscillating waves or coherent structures that are outlined below.

Theorem 1. *[global and long time dynamics of charged particles in axisymmetric devices] For axisymmetric configurations like in tokamaks, consider the standard model (2.12) used to model the exterior magnetic field $\mathbf{B}(x)$. Suppose that the self-consistent electromagnetic field $(\mathbf{E},\mathbf{B})(\tau,x)$ is a smooth known given function, and that the Assumption 3.2 given in Paragraph 3.2.3 is satisfied. Then, the phase space may be decomposed into a finite number of disjoint open subsets Ω_j such that:*

$$(1.5)\qquad \bar\Omega_0\cup\bar\Omega_1\cup\cdots\cup\bar\Omega_m=\mathbb{R}^3\times\mathbb{R}^3\,,\qquad \Omega_m\neq\emptyset\,,\qquad \mathring{\bar\Omega}_j=\Omega_j\,,\qquad m\in\mathbb{N}\,.$$

Fix any $j\in\{0,\cdots,m\}$ and any compact subset $\mathrm{K}\Subset\Omega_j$. Uniformly in $(x_0,v_0)\in\mathrm{K}$, there is some $\varepsilon_0\in]0,1[$, a time $\mathcal{T}\in\mathbb{R}_+^*$ and profiles $(X_j,V_j)(\tau,x,v,s,\upsilon)\in\mathcal{C}^\infty([0,\mathcal{T}]\times\mathrm{K}\times\mathbb{T}^2;\mathbb{R}^6)$ with $j\in\mathbb{N}$ such that the asymptotic behaviour when $\varepsilon\in]0,\varepsilon_0[$ goes to zero of the solution $(x_\varepsilon,v_\varepsilon)(\tau,x_0,v_0)$ to (1.3) can be approximated with infinite accuracy in the sup-norm by the following multiscale and multiphase expansion:

$$(1.6)\qquad \begin{pmatrix}x_\varepsilon\\ v_\varepsilon\end{pmatrix}(\tau,x_0,v_0)\underset{\varepsilon\to0}{\sim}\sum_{j=0}^{\infty}\varepsilon^j\begin{pmatrix}X_j\\ V_j\end{pmatrix}\Big(\tau,x_0,v_0,\frac{\psi_l(\tau,x_0,v_0)}{\varepsilon},\frac{\psi_{s\varepsilon}(\tau,x_0,v_0)}{\varepsilon^2}\Big)\,.$$

The profiles ${}^t(X_j,V_j)$ and the phases $(\psi_l,\psi_{s\varepsilon})$ are determined by easily solvable equations. The phase $\psi_{s\varepsilon}(\cdot)$ is itself an oscillation of small amplitude as indicated in (4.59).

The above oscillating structures are detailed in Paragraph 4.2.2. We find $\partial_\upsilon X_0\equiv0$ and $\partial_s X_0\not\equiv0$, whereas $\partial_\upsilon V_0\not\equiv0$ and $\partial_s V_0\not\equiv0$. Thus, the spatial part x_ε of (1.6) involves large amplitude oscillations related to ψ_l as well as, with the higher frequency ε^{-2}, small amplitude oscillations with respect to $\psi_{s\varepsilon}$. The derivatives of ${}^t(x_\varepsilon,v_\varepsilon)$ are of the order ε^{-2} or ε^{-1}, which corresponds to the production of steep gradients. All these contributions are physically significant because they all carry a large amount of energy.

The Vlasov equation (1.1) can be associated with an initial data $f_0(\cdot)$ or a source term $g(\cdot)$. Then, the distribution $f(\tau,\cdot)$ is obtained by solving the characteristics backwards. If $f_0(\cdot)$ or $g(\cdot)$ are not well prepared, that is if they are not functions of the adiabatic invariants, the oscillations of (1.6) are triggered. Theorem 1 is therefore essential to understand plasmas which are in a state far from thermodynamic equilibrium. As explained in Paragraph 4.1.2, it furnishes concrete dynamical criteria (which depend on the choice of Ω_j) for the long time confinement of plasmas. It is a key to explore research fields like the non-diffusive transport or, in the spirit of [13], the intermittency of electromagnetic waves. It could also help develop computational methods or explain the origin of many microinstabilities.

The structure of the paper is as follows: Section 2 furnishes useful tools to deal with the axisymmetric geometrical framework; Section 3 is designed to filter the oscillations; and the last Section 4 is devoted to the WKB calculus and the justification of (1.6).

2. Axisymmetric toroidal configurations.

The axisymmetric configurations furnish a paradigm for all types of magnetic confinement systems [3, 24]. To study toroidal devices like tokamaks and spheromax, it is better to work with a phase space that is adapted to the underlying dynamics. Following [7, 10], this entails a change of spatial coordinates, which will be achieved in a two-step approach. Starting from the cartesian coordinates x, this leads in Subsection 2.1 to the *toroidal coordinates* T, and then in Subsection 2.2 to the *flux coordinates* F. Expressed in terms of F, the field lines are still not straight. As explained in Subsection 2.3, this can be remedied by decomposing the velocity field in an adequate frame field. Subsection 2.4 gives an overview of these transformations. In connection with the forthcoming analysis, some *geometrical features* must be displayed. The related aspects are clarified in Subsection 2.5.

2.1. **Toroidal coordinates.** In dimensionless units (Paragraph 3.1.1), a position $x = \tilde{\mathrm{x}}/\mathrm{r}$ (with $\mathrm{r} \in \mathbb{R}_+^*$ to be specified later) can be traced by its cartesian coordinates:

$$x = \frac{\tilde{\mathrm{x}}}{\mathrm{r}} = x^1\, \mathfrak{e}_1^c + x^2\, \mathfrak{e}_2^c + x^3\, \mathfrak{e}_3^c\,, \qquad \mathfrak{e}_1^c := \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \qquad \mathfrak{e}_2^c := \begin{pmatrix} 0\\1\\0 \end{pmatrix}, \qquad \mathfrak{e}_3^c := \begin{pmatrix} 0\\0\\1 \end{pmatrix}. \tag{2.1}$$

The same point can also be identified by a set of one radial coordinate $r \in \mathbb{R}_+$ (the distance from the magnetic axis) and two angle coordinates $\theta \in \mathbb{T} := \mathbb{R}/(2\,\pi\,\mathbb{Z})$ (the poloidal angle) and $\phi \in \mathbb{T}$ (the toroidal angle) which reflect the rotational invariances. These choices lead to the selection of the *toroidal coordinates*:

$(r, \theta, \phi) \in [0, r_0] \times \mathbb{T}^2$.

$R_0 \in \mathbb{R}_+^*$ is the major radius.

$r_0 \in \mathbb{R}_+^*$ is the minor radius.

The distance from the major axis of the torus is:

$R = \mathrm{R}(r, \theta) := R_0 + r\,\cos\theta$.

Change of coordinates:

$$x = \begin{pmatrix} \mathrm{R}\,\cos\phi \\ \mathrm{R}\,\sin\phi \\ r\,\sin\theta \end{pmatrix} \underset{\Sigma_c^t}{\overset{\Sigma_t^c}{\leftrightarrows}} \mathit{T} = \begin{pmatrix} r\\ \theta\\ \phi \end{pmatrix}.$$

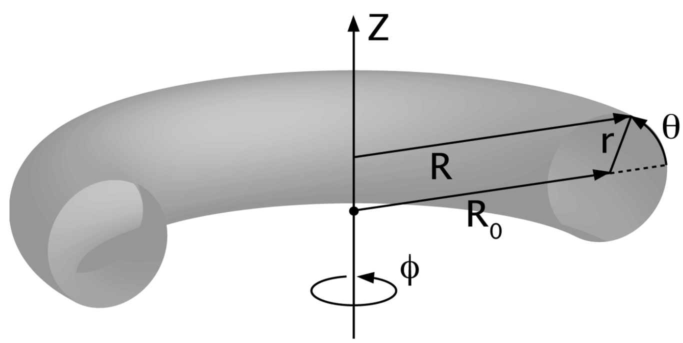


Figure 1. Toroidal coordinates.

The new curvilinear coordinates r, θ and ϕ generate a reciprocal basis. The corresponding normalized basis $(\mathsf{e}_r, \mathsf{e}_\theta, \mathsf{e}_\phi)$ is orthonormal. It is such that:

$$\mathsf{e}_r(\mathit{T}) = \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ \sin\theta \end{pmatrix}, \qquad \mathsf{e}_\theta(\mathit{T}) := \begin{pmatrix} -\sin\theta\,\cos\phi \\ -\sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathsf{e}_\phi(\mathit{T}) := \begin{pmatrix} +\sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}. \tag{2.2}$$

The plasma is supposed to be confined inside the (compact) toroidal chamber $\mathscr{M}$ which is:

$$\mathscr{M} := \{\, (r, \theta, \phi) \in \mathbb{R}_+ \times \mathbb{T}^2\,;\, r \le r_0 \,\}\,. \tag{2.3}$$

2.2. **Flux coordinates.** For axisymmetric systems, the field lines lie in nested magnetic flux surfaces, say $\mathscr{M}_\psi$ with $\psi \in \mathbb{R}_+$. The cuts of these flux surfaces with the plane which passes through the $z-$axis are closed curves which can be viewed as the level sets of a *poloidal flux function* $\uppsi(\cdot)$. Hence, the *poloidal cross sections* $\mathscr{C}_\psi$ take the following form:

$$\mathscr{C}_\psi = \mathscr{M}_\psi \cap \big\{(r,\theta,\phi);\phi=0\big\} = \big\{\,(r,\theta)\in\mathbb{R}_+\times\mathbb{T}\,;\,\uppsi(r,\theta)=\psi\,\big\}\,,\qquad \psi\in\mathbb{R}_+\,. \tag{2.4}$$

In practice, the function $\uppsi(\cdot,\theta)$ is only defined for $r \leq r_0$. But, to avoid technicalities, we will consider that it is globally defined.

Assumption 2.1. *[structure of the poloidal cross sections]*

$$\uppsi(\cdot)\in\mathcal{C}^\infty(\mathbb{R}_+\times\mathbb{T};\mathbb{R})\,, \tag{2.5a}$$

$$\uppsi(0,\theta)=0\,,\qquad \forall\,\theta\in\mathbb{T}\,, \tag{2.5b}$$

$$\partial_r\uppsi(r,\theta)>0\,,\qquad \forall\,(r,\theta)\in\mathbb{R}_+^*\times\mathbb{T}\,, \tag{2.5c}$$

$$\lim_{r\to+\infty}\uppsi(r,\theta)=+\infty\,,\qquad \forall\,\theta\in\mathbb{T}\,. \tag{2.5d}$$

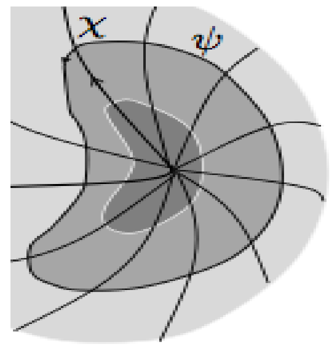


Figure 2. Flux coordinates.

It follows from (2.5) that the poloidal cross-sections $\mathscr{C}_\psi$ are diffeomorphic to concentric circles. We can find orthogonal trajectories to the family of curves $\mathscr{C}_\psi$ which may be found as the level curves of a new function $\upchi(\cdot)$ satisfying:

$$\upchi(r,\theta+2\,\pi)=\upchi(r,\theta)+2\,\pi\,,\qquad \partial_\theta\upchi(r,\theta)>0\,,\qquad \forall\,(r,\theta)\in\mathbb{R}_+^*\times\mathbb{T}\,, \tag{2.6a}$$

$$\nabla_p\upchi\cdot\nabla_p\uppsi=\partial_r\upchi\,\partial_r\uppsi+r^{-2}\,\partial_\theta\upchi\,\partial_\theta\uppsi=0\,,\qquad \forall\,(r,\theta)\in\mathbb{R}_+^*\times\mathbb{T}\,, \tag{2.6b}$$

where $\nabla_p=\partial_r\,\mathsf{e}_r+r^{-1}\,\partial_\theta\,\mathsf{e}_\theta$ stands for the gradient in the polar coordinates (r,θ). With the identification (2.6a), the values $\chi\in\mathbb{T}$ of $\upchi(\cdot)$ can be viewed as angles. The flux coordinates are defined by $F:={}^t(\psi,\chi)\in\mathbb{R}_+\times\mathbb{T}$. They give access to the curvilinear coordinates:

$$\begin{pmatrix}\psi\\ \chi\\ \phi\end{pmatrix}\equiv\begin{pmatrix}\uppsi(r,\theta)\\ \upchi(r,\theta)\\ \upphi(\phi)\end{pmatrix}\underset{\Sigma_f^t}{\overset{\Sigma_t^f}{\leftrightarrows}} T=\begin{pmatrix}r\\ \theta\\ \phi\end{pmatrix},\qquad \begin{pmatrix}\psi\\ \chi\\ \phi\end{pmatrix}\underset{\Sigma_f^c}{\overset{\Sigma_c^f}{\leftrightarrows}} x\,,\qquad \begin{array}{l}\Sigma_c^f:=\Sigma_t^f\circ\Sigma_c^t\,,\\[1ex] \Sigma_f^c:=\Sigma_t^c\circ\Sigma_f^t\,.\end{array} \tag{2.7}$$

Remark that:

$$\Sigma_f^t(F,\phi)=\big(\check\Sigma_f^t(F),\phi\big)\,,\qquad F=\check\Sigma_t^f(r,\theta):={}^t(\uppsi,\upchi)(r,\theta)\,,\qquad \check\Sigma_f^t:=(\check\Sigma_t^f)^{-1}\,. \tag{2.8}$$

Define the function $\varpi(\cdot)$ which represents the angle of the direction $\nabla_p\uppsi$ in the basis (e_r,e_θ) :

$$\begin{array}{rcl}\varpi:\mathbb{R}_+\times\mathbb{T} & \longrightarrow &]-\pi/2,+\pi/2\,[\\ F=(\psi,\chi) & \longmapsto & \varpi=\varpi(F):=\arctan\big(r^{-1}\,\partial_r\uppsi^{-1}\,\partial_\theta\uppsi\big)\big(\check\Sigma_f^t(F)\big)\,.\end{array} \tag{2.9}$$

It is adjusted such that $e_\phi(F,\phi):=\mathsf{e}_\phi\circ\Sigma_f^t(F,\phi)=\mathsf{e}_\phi(\phi)$ and:

$$e_\psi(F,\phi):=\frac{{}^t\nabla_p\uppsi}{|\nabla_p\uppsi|}=+\cos\varpi(F)\;\mathsf{e}_r\circ\Sigma_f^t(F,\phi)+\sin\varpi(F)\;\mathsf{e}_\theta\circ\Sigma_f^t(F,\phi)\,, \tag{2.10a}$$

$$e_\chi(F,\phi):=\frac{{}^t\nabla_p\upchi}{|\nabla_p\upchi|}=-\sin\varpi(F)\;\mathsf{e}_r\circ\Sigma_f^t(F,\phi)+\cos\varpi(F)\;\mathsf{e}_\theta\circ\Sigma_f^t(F,\phi)\,. \tag{2.10b}$$

A basic example is when the poloidal cross sections form concentric circles. This situation is a bit unrealistic, and it does not exploit the full potential of the functions $\psi(\cdot)$ and $\chi(\cdot)$. But, it will serve as guidelines for the testing of Assumptions and results. This is why it will appear repeatedly throughout the text.

Example 1. *[toy model] The choices $\psi(r,\theta)=r$ and $\chi(r,\theta)=\theta$ are compatible with (2.5) and (2.6). Then $F={}^t(r,\theta)$ and $T\equiv(F,\phi)$. Moreover:*

$$\check{\Sigma}_t^f(r,\theta)={}^t(r,\theta)\,,\qquad \varpi\equiv 0\,,\qquad e_\psi\equiv \mathsf{e}_r\,,\qquad e_\chi\equiv \mathsf{e}_\theta\,. \tag{2.11}$$

The basis $(e_\psi,e_\chi,e_\phi)(F,\phi)$ is clearly orthonormal and positively oriented. The introduction of F facilitates the spatial description of axisymmetric devices. But it does not take into account the entire information contained in $\mathbf{B}$, because it says nothing about the directions inside the cotangent bundle $T\mathscr{M}$ pointed to by the external magnetic field.

2.3. **Magnetic coordinates.** The field lines are generated by a magnetic field having the form $\mathrm{b}_0\,\mathbf{B}(\tilde{\mathrm{x}}/\mathrm{r})$, where b_0 is a typical size of the amplitude of the magnetic field. The present study starts from the standard descriptions of $\mathbf{B}(x)$. For instance, we can refer to the book [10], chapter 3.6. The vector field $\mathbf{B}(\cdot)$ can be decomposed according to:

$$\mathbf{B}=\mathbf{B}_{tor}+\mathbf{B}_{pol}\,,\qquad \mathbf{B}_{tor}:=I\;\nabla\phi\,,\qquad \mathbf{B}_{pol}:=\nabla\phi\times\nabla\psi\,,\qquad \mathbf{B}_{tor}\perp\mathbf{B}_{pol}\,. \tag{2.12}$$

The toroidal component $\mathbf{B}_{tor}$ is produced by electromagnets. The axisymmetric hypothesis implies that $\partial_\phi I\equiv 0$. Moreover, for isotropic pressure plasmas, the poloidal current I does not depend on χ so that $I\equiv \mathrm{I}(\psi)$, where $\mathrm{I}:\mathbb{R}_+\longrightarrow\mathbb{R}_+^*$ is a smooth positive function. The poloidal component $\mathbf{B}_{pol}$ is supposed to be the result of the toroidal electric current that flows inside the plasma. The magnetic field $\mathbf{B}(\cdot)$ is nowhere-vanishing, with amplitude:

$$0<b:=\big(b_{tor}^2+b_{pol}^2\big)^{1/2}\,,\qquad b_{tor}:=I\;|\nabla\phi|\,,\qquad b_{pol}:=|\nabla\phi|\;|\nabla\psi|\,. \tag{2.13}$$

Thus, expressed in flux coordinates, the scalar functions $b(\cdot)$, $b_{tor}(\cdot)$, and $b_{pol}(\cdot)$ are just:

$$b(F)=(\mathrm{R}^{-1}\,a)\circ\check{\Sigma}_f^t(F)\,,\qquad\qquad a(r,\theta):=\big[r^{-2}\,(\partial_\theta\psi)^2+(\partial_r\psi)^2+I(\psi)^2\big]^{1/2}\,, \tag{2.14a}$$

$$0<b_{tor}(F)=\mathrm{I}(\psi)\;(\mathrm{R}^{-1})\circ\check{\Sigma}_f^t(F)\,, \tag{2.14b}$$

$$0<b_{pol}(F)=(\mathrm{R}^{-1}\,|\nabla_p\psi|)\circ\check{\Sigma}_f^t(F)\,. \tag{2.14c}$$

Introduce the function $\omega(\cdot)$ representing the angle of the direction $\mathbf{B}(\cdot)$ in the basis (e_χ,e_ϕ) :

$$\begin{aligned} \omega:\mathbb{R}_+\times\mathbb{T}\quad&\longrightarrow\quad]0,+\pi/2[\\ F=(\psi,\chi)\quad&\longmapsto\quad \omega=\omega(F):=\arctan\,(b_{tor}^{-1}\;b_{pol})(F)\,. \end{aligned} \tag{2.15}$$

The unitary vector field $e_\parallel(\cdot)$ which is generated by the directions of $\mathbf{B}(\cdot)$ is given by:

$$e_\parallel(F,\phi):=\sin\omega(F)\;e_\chi(F,\phi)+\cos\omega(F)\;e_\phi(F,\phi)\,,\qquad \sin\omega=b^{-1}\,b_{pol}>0\,. \tag{2.16}$$

Expressed in toroidal coordinates, this becomes:

$$e_\parallel\circ\Sigma_t^f(T):=-\,\frac{\partial_\theta\psi}{r\;a}\;\mathsf{e}_r(T)+\frac{\partial_r\psi}{a}\;\mathsf{e}_\theta(T)+\frac{I(\psi)}{a}\;\mathsf{e}_\phi(T)\,. \tag{2.17}$$

The unitary direction $e_\parallel$ can be completed in order to obtain some positively oriented (non holonomic) orthonormal basis $(e_\psi, e_\perp, e_\parallel)$. It suffices to take $e_\psi \equiv e_\psi$ and:

$$e_\perp(\textsc{f},\phi) := (e_\parallel \times e_\psi)(\textsc{f},\phi) = \cos\omega(\textsc{f})\; e_\chi(\textsc{f},\phi) - \sin\omega(\textsc{f})\; e_\phi(\textsc{f},\phi)\,. \tag{2.18}$$

Example 2. *[toy model] With $\psi(r,\theta) = r$ and $\chi(r,\theta) = \theta$, we simply find:*

$$\omega(\textsc{f}) \equiv \omega(\textsc{t}) \equiv \omega(r) = \arctan\, I(r)^{-1}\,, \qquad a(r,\theta) \equiv a(r) = \left[1 + I(r)^2\right]^{1/2}. \tag{2.19}$$

The vector field $\mathbf{B}(\cdot)$ remains contained in the plane $\langle e_\chi, e_\phi\rangle$. Viewed as a function of $(\textsc{f},\phi)$ and projected in the moving plane (e_χ, e_ϕ), it is still inhomogeneous with possible variations in amplitudes (through the function b) and directions (through the function ω). To straighten out the field lines, we can further implement the Clebsch representation, as indicated in [10]. Unfortunately, the associated magnetic coordinates $\textsc{m}$ are usually not orthogonal. As a matter of fact, they allow to catch the direction $e_\parallel$, but they generate a contravariant curvilinear basis which has little to do with $(e_\psi, e_\perp, e_\parallel)$. The advantages resulting from the use of Clebsch variables are not clear when looking at the dynamics of charged particles. In particular, the euclidean norm of the components of $d\textsc{m}/dt$ would not be associated with the kinetic energy, and therefore it would not be conserved. For this reason, the transition from $\textsc{f}$ to $\textsc{m}$ will not be adopted. We will proceed differently by decomposing the velocity field v in the frame field $(e_\psi, e_\perp, e_\parallel)$.

2.4. **Overview of the changes of basis.** The aim of this Subsection 2.4 is to put apart some notations and relations that will appear repeatedly in the text. The letters c, t, f and m refer respectively to <u>c</u>anonical, <u>t</u>oroidal, $f$lux and <u>m</u>agnetic representations. The transformation allowing to pass from the coordinates of type $\star \in \{c,t,f\}$ to the coordinates of type $* \in \{c,t,f\}$ is denoted by $\Sigma^*_\star$. We have $\Sigma^\star_\star \equiv Id$ for all $\star \in \{c,t,f\}$. Moreover, in accordance with Subsections 2.1 and 2.2, we have:

$$\Sigma^t_c(x) = \textsc{t} = \begin{pmatrix} \mathrm{r} \\ \theta \\ \phi \end{pmatrix}(x)\,, \qquad \Sigma^f_t(\textsc{t}) = \textsc{f} = \begin{pmatrix} \psi \\ \chi \\ \phi \end{pmatrix}(\textsc{t})\,, \qquad \Sigma^f_c(x) = \Sigma^f_t \circ \Sigma^t_c(x)\,.$$

The styles r, θ, ϕ, ψ and χ will be preferentially involved when r, θ, ϕ, ψ and χ are viewed as functions of the preceding variables. Consider the unitary vector fields:

$$e^*_{\star i} := \frac{{}^t\nabla_\star \Sigma^{*i}_\star}{|\nabla_\star \Sigma^{*i}_\star|}\,, \qquad e^*_{\star\diamond i} := e^*_{\star i} \circ \Sigma^\star_\diamond\,, \qquad \Sigma^*_\star = \begin{pmatrix} \Sigma^{*1}_\star \\ \Sigma^{*2}_\star \\ \Sigma^{*3}_\star \end{pmatrix}, \qquad (*,\star,\diamond) \in \{c,t,f\}^3. \tag{2.20}$$

This gives rise to orthonormal basis:

$$(e^*_{\star 1}, e^*_{\star 2}, e^*_{\star 3})\,, \qquad (e^*_{\star\diamond 1}, e^*_{\star\diamond 2}, e^*_{\star\diamond 3}), \qquad (*,\star,\diamond) \in \{c,t,f\}\,. \tag{2.21}$$

Another way to formulate (2.20) is to say that $e^*_{\star i}$ is the i^{th} component of the orthonormal basis associated to the coordinate system of type $*$, as it can be further decomposed in the orthonormal basis coming from the initial coordinate system of type $\star$, allowing us to talk about the j^{th} component of $e^*_{\star i}$ in the basis $(e^\star_{\star 1}, e^\star_{\star 2}, e^\star_{\star 3})$. The vector $e^*_{\star\diamond i}$ is just $e^*_{\star i}$ viewed as a function of the variable of type $\diamond$.

In fact, the last definitions can also be applied in the case $* = m$, yielding for instance:

$$e^m_{f1}(F) := e_\psi(F)\,, \qquad e^m_{f2}(F) := e_\perp(F)\,, \qquad e^m_{f3}(F) := e_\parallel(F)\,. \tag{2.22}$$

The various possible changes of coordinates are summarized in the table below.

	cartesian	**toroidal**	**flux**	**magnetic**
curvilinear coordinates	(x^1,x^2,x^3)	(T^1,T^2,T^3) $=(r,\theta,\phi)$	(F^1,F^2,F^3) $=(\psi,\chi,\phi)$	(M^1,M^2,M^3)
orthonormal basis	$(\mathfrak{e}^c_1,\mathfrak{e}^c_2,\mathfrak{e}^c_3)$	$(\mathsf{e}^t_{ct1},\mathsf{e}^t_{ct2},\mathsf{e}^t_{ct3})$ $=(\mathsf{e}_r,\mathsf{e}_\theta,\mathsf{e}_\phi)$	$(\mathsf{e}^f_{tf1},\mathsf{e}^f_{tf2},\mathsf{e}^f_{tf3})$ $=(\mathsf{e}_\psi,\mathsf{e}_\chi,\mathsf{e}_\phi)$	$(e^m_{f1},e^m_{f2},e^m_{f3})$ $=(e_\psi,e_\perp,e_\parallel)$

Figure 3. Various changes of coordinates.

By convention, for $\star\in\{c,t,f\}$ and for $*\in\{c,t,f,m\}$, the orthonormal matrix $\mathrm{O}^*_\star$ makes the transition from the orthonormal basis $(e^*_{\star1},e^*_{\star2},e^*_{\star3})$ to the canonical basis associated to the coordinates of type $\star$, that is $(e^\star_{\star1},e^\star_{\star2},e^\star_{\star3})$. In other words:

$${}^t\mathrm{O}^*_\star = (e^*_{\star1},e^*_{\star2},e^*_{\star3}) = (\mathrm{O}^*_\star)^{-1}\,, \qquad \mathrm{O}^*_\star\, e^*_{\star i} = e^\star_{\star i}\,, \qquad \mathrm{O}^*_{\star\diamond} := \mathrm{O}^*_\star\circ\Sigma^\star_\diamond\,. \tag{2.23}$$

In (2.23), the matrix ${}^t\mathrm{O}^*_\star$ is written in column vectors. With this in mind, observe that:

$$\mathrm{O}^*_\star = (\mathrm{O}^\star_{**})^{-1} = {}^t\mathrm{O}^\star_{**}\,, \qquad \mathrm{O}^\triangleleft_{\star\diamond}\,\mathrm{O}^*_{\triangleleft\diamond} = \mathrm{O}^*_{\star\diamond}\,. \tag{2.24}$$

Recall also that $e^*_{\star i}$ is decomposed in the orthonormal basis associated to the coordinates of type $\star$. From the preceding definitions, it is clear that the functions $e^*_\star$, $e^*_{\star i}$ and $\mathrm{O}^*_\star$ are considered as depending on the coordinates of type $\star$. On the contrary, the functions $e^*_{\star\diamond}$, $e^*_{\star\diamond i}$ and $\mathrm{O}^*_{\star\diamond}$ depend on the coordinates of type $\diamond$. In fact, the matrices $\mathrm{O}^f_{tf}(\cdot)$ and $\mathrm{O}^m_f(\cdot)$ are only functions of F, and they can be identified with the planar rotations:

$$\mathrm{O}^f_{tf}(F) := \begin{pmatrix} +\cos\varpi & \sin\varpi & 0\\ -\sin\varpi & \cos\varpi & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad \mathrm{O}^m_f(F) := \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\omega & -\sin\omega\\ 0 & \sin\omega & +\cos\omega\end{pmatrix}. \tag{2.25}$$

By construction, for all $(\star,*)\in\{c,t,f\}^2$ (the choice $*=m$ being excluded), we have:

$$d_\star\Sigma^*_\star = \mathrm{D}^*_\star\ \mathrm{O}^*_\star\,, \qquad \mathrm{D}^*_\star = (\mathrm{D}^*_{\star i}\,\delta_{ij})_{ij}\,, \qquad \mathrm{D}^*_{\star\diamond} = (\mathrm{D}^*_{\star\diamond i}\,\delta_{ij})_{ij} := \mathrm{D}^*_\star\circ\Sigma^\star_\diamond\,, \tag{2.26}$$

where $d_\star$ marks the differential with respect to the variable of type $\star$, whereas $\mathrm{D}^*_\star$ is a diagonal matrix with elements $\mathrm{D}^*_{\star i}$. The coefficients $\mathrm{D}^*_{\star i}$ can be interpreted as the inverses of Lamé coefficients. As a matter of fact, they come from the amplitudes of the gradients of the curvilinear coordinates. Of particular interest will be the following coefficients:

$$\mathrm{D}^f_{cf1}(F) := \big[\,\partial_r\psi(r,\theta)^2 + r^{-2}\,\partial_\theta\psi(r,\theta)^2\,\big]^{1/2}_{|(r,\theta)=\check\Sigma^t_f(F)}\,, \tag{2.27a}$$

$$\mathrm{D}^f_{cf2}(F) := \big[\,\partial_r\chi(r,\theta)^2 + r^{-2}\,\partial_\theta\chi(r,\theta)^2\,\big]^{1/2}_{|(r,\theta)=\check\Sigma^t_f(F)}\,, \tag{2.27b}$$

$$\mathrm{D}^f_{cf3}(F) := \mathrm{R}\circ\check\Sigma^t_f(F)^{-1} = (R_0 + r\,\cos\theta)^{-1}_{|(r,\theta)=\check\Sigma^t_f(F)}\,. \tag{2.27c}$$

Example 3. *[toy model] With $\psi(r,\theta) = r$ and $\chi(r,\theta)=\theta$, we simply find:*

$$\mathrm{D}^{f}_{cf1}(\tau) = 1\,, \qquad \mathrm{D}^{f}_{cf2}(\tau) = r^{-1}\,, \qquad \mathrm{D}^{f}_{cf3}(\tau) = \mathrm{R}^{-1}\,. \tag{2.28}$$

2.5. **Geometrical settings.** The circle group $\mathbb{T} := \mathbb{R}/(2\,\pi\,\mathbb{Z})$ has a transitive and faithful action on the plane perpendicular to the direction $e^m_{fc3}(x) = e_{\parallel} \circ \Sigma^f_c(x)$, that is:

$$e^m_{fc3}(x)^{\perp} \equiv e_{\parallel} \circ \Sigma^f_c(x)^{\perp} := \{\, v \in \mathbb{R}^3\,;\ v \cdot e^m_{fc3}(x) = 0 \,\}\,.$$

Starting from $(e^m_{fc1}, e^m_{fc2})(x)$, it allows to reach all other orthonormal basis of $(e^m_{fc3})(x)^{\perp}$. To this end, it suffices to define:

$$(\upsilon * e^m_{fc1}, \upsilon * e^m_{fc2}) := (\cos\upsilon\, e^m_{fc1} + \sin\upsilon\, e^m_{fc2}, -\sin\upsilon\, e^m_{fc1} + \cos\upsilon\, e^m_{fc2})\,, \qquad \upsilon \in \mathbb{T}\,. \tag{2.29}$$

By convention, for all $\upsilon \in \mathbb{T}$, we set $\upsilon * e^m_{fc3} := e^m_{fc3}$. Consider the Ricci rotation coefficients which are associated to the selection of $(\upsilon * e^m_{fc1}, \upsilon * e^m_{fc2})$. In other words:

$$\Gamma^{cj}_{3i}(x,\upsilon) := (\upsilon * e^m_{fci}(x) \cdot \nabla_x)^t e^m_{c3}(x) \cdot (\upsilon * e^m_{fcj}(x))\,, \qquad (i,j) \in \{1,2,3\}^2\,. \tag{2.30}$$

When looking at (2.30), it may be helpful to take into account the relations:

$$(e^m_{fc\star} \cdot \nabla_x)^t e^m_{fci} \cdot e^m_{fci} = 0\,, \qquad (e^m_{fc\star} \cdot \nabla_x)^t e^m_{fci} \cdot e^m_{fcj} + (e^m_{fc\star} \cdot \nabla_x)^t e^m_{fcj} \cdot e^m_{fci} = 0\,. \tag{2.31}$$

Define:

$$\Gamma^{*j}_{3i}(\cdot,\upsilon) := \Gamma^{cj}_{3i}(\Sigma^c_*(\cdot),\upsilon)\,, \qquad (i,j) \in \{1,2,3\}^2\,, \qquad * \in \{t,f\} \tag{2.32}$$

For $(i,j) \in \{1,2\}^2$, the definition of $\Gamma^{cj}_{3i}(x,\cdot)$ is based on $(e^m_{fc1}, e^m_{fc2})(x)$. As a matter of fact, it is linked with the choice inside the plane $e^m_{fc3}(x)^{\perp}$ of the special basis $(e^m_{fc1}, e^m_{fc2})(x)$. On the contrary, the Fourier coefficients:

$$\mathfrak{F}\big(\Gamma^{cj}_{3i}(x,\cdot)\big)_k := \frac{1}{2\,\pi} \int_0^{2\,\pi} e^{-i\,k\,\upsilon}\ \Gamma^{cj}_{3i}(x,\upsilon)\ d\upsilon\,, \qquad k \in \mathbb{Z} \tag{2.33}$$

are expressions which depend only on $e^m_{fc3}(\cdot)$. In particular, we can put aside:

$$\mathfrak{F}(\Gamma^{cj}_{33})_0 = \mathfrak{F}(\Gamma^{c3}_{3i})_0 = 0\,, \qquad \forall\,(i,j) \in \{1,2,3\}^2\,, \tag{2.34a}$$

$$\mathcal{C} := \mathfrak{F}(\Gamma^{c1}_{31})_0 = \mathfrak{F}(\Gamma^{c2}_{32})_0 = \frac{1}{2}\ [\,(e^m_{fc1} \cdot \nabla_x)^t e^m_{fc3} \cdot e^m_{fc1} + (e^m_{fc2} \cdot \nabla_x)^t e^m_{fc3} \cdot e^m_{fc2}\,]\,, \tag{2.34b}$$

We can also specify how the geometrical quantity $\mathcal{C}$ is related to I and ψ.

Lemma 2.1. *[computation of $\mathcal{C}$] With $a(\cdot)$ as in (2.14), we have:*

$$\mathcal{C}(x) = \frac{1}{2}\ \mathrm{div}\, e^m_{fc3}(x) = \frac{1}{2\,r}\ \Big[-\partial_r\Big(\frac{\partial_\theta \psi}{a}\Big) + \partial_\theta\Big(\frac{\partial_r \psi}{a}\Big) \Big] (\Sigma^t_c(x))\,. \tag{2.35a}$$

Proof. Since $(e^m_{fc3} \cdot \nabla_x)^t e^m_{fc3} \cdot e^m_{fc3} = 0$, we can interpret $\mathcal{C}$ according to:

$$\mathcal{C} = \frac{1}{2}\ \sum_{j=1}^{3} (e^m_{fcj} \cdot \nabla_x)^t e^m_{fc3} \cdot e^m_{fcj} \ = \ \frac{1}{2}\ \mathrm{Tr}\ d_x e^m_{fc3} \ = \ \frac{1}{2}\ \mathrm{div}\, e^m_{fc3} \ = \ \frac{1}{2}\ (\mathrm{div}_T\, e_{\parallel} \circ \Sigma^f_t) \circ \Sigma^t_c\,.$$

Look at (2.17). This gives immediately access to (2.35a). $\square$

It follows from (2.5) that, for all $\theta \in \mathbb{T}$, the function $\psi(\cdot,\theta) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is bijective. The inverse function of $\psi(\cdot,\theta)$ is denoted by $\pitchfork(\cdot,\theta)$. It is such that:

$$\begin{array}{rcl} \pitchfork : \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ \psi & \longmapsto & \pitchfork(\psi,\theta)\,, \end{array} \qquad \pitchfork(\psi(r,\theta),\theta) = r\,, \qquad \psi(\pitchfork(\psi,\theta),\theta) = \psi\,. \tag{2.36}$$

Introduce the parametrization $\gamma_\psi : \mathbb{T} \longrightarrow \mathbb{R}_+ \times \mathbb{T}$ given by:

$$\gamma_\psi(\theta) := \begin{pmatrix} \pitchfork(\psi,\theta) \\ \theta \end{pmatrix}, \qquad \gamma'_\psi(\theta) = \frac{1}{\partial_r \psi(\psi,\theta)} \begin{pmatrix} -\partial_\theta \psi(\psi,\theta) \\ +\partial_r \psi(\psi,\theta) \end{pmatrix}. \tag{2.37}$$

The level set $\mathscr{C}_\psi$ of (2.4) can be parametrized through $\mathscr{C}_\psi = \{\gamma_\psi(\theta)\,;\,\theta \in \mathbb{T}\} \subset \mathbb{R}_+ \times \mathbb{T}$.

Lemma 2.2. *[circulation around $\mathscr{C}_\psi$] For all $\psi \in \mathbb{R}_+^*$ and all $F \in \mathcal{C}^0(\mathbb{R}_+^*;\mathbb{R})$, we have:*

$$\int_0^{2\pi} \Big[\frac{r\, \mathcal{C}_t\, F(a)}{\partial_r \psi} \Big] \big(\gamma_\psi(\theta)\big)\, d\theta = 0\,, \qquad \mathcal{C}_\star := \mathcal{C} \circ \Sigma_\star^c\,. \tag{2.38}$$

Proof. Let $a \mapsto G(a)$ be a primitive of $a \mapsto a^{-2}\, F(a)$. The line integral over the closed curve $\mathscr{C}_\psi$ of the vector field generated by the potential $G \circ a$ is zero, meaning that:

$$0 = \int_{\mathscr{C}_\psi} \nabla(G\circ a)\cdot dl = \int_0^{2\pi} \big[a^{-2}\, F(a)\big]\big(\gamma_\psi(\theta)\big)\ {}^t(\partial_r a, \partial_\theta a)\big(\gamma_\psi(\theta)\big)\cdot \gamma'_\psi(\theta)\ d\theta\,. \tag{2.39}$$

Then, exploit (2.35a) and (2.37). By this way, we can easily extract (2.38). □

Example 4. *[toy model] The formula (2.35) gives rise to $\mathcal{C} \equiv 0$.*

The properties of $\mathcal{C}$ play an important role in the subsequent analysis. It is therefore worth mentioning that the current situation about $\mathcal{C}$ differs completely from [13]. In [13], the function $\mathcal{C}(\cdot)$ was strictly monotone along the field lines with a unique change of sign. By contrast, in view of (2.38) applied with $F \equiv 1$ and taking into account (2.5c), we see that the function $\mathcal{C}(\cdot)$ of (2.34b) is bounded, and that it involves at least two changes of sign along $\mathscr{C}_\psi$. It can even be zero on open parts of $\mathscr{C}_\psi$, where $\psi(\cdot)$ does not depend on θ. For instance, in Example 1, we simply find $\mathcal{C} \equiv 0$.

3. Reformulation of the dynamics.

Before getting to times $\tau \simeq 1$, the solution (x,v) to (1.3) undergoes two different types of rapid oscillations. The purpose of this Section 3 is to properly separate the two underlying time-scales . To catch the oscillations appearing first, it is better to change τ into what will be called the *intermediate times* $t := \varepsilon^{-1}\,\tau$. Then, the system (1.3) becomes:

$$\left\{ \begin{array}{rcll} \partial_t x & = & v\,, & x(0) = x_0\,, \\ \partial_t v & = & \varepsilon^{-1}\,\mathrm{c}(|v|)\,\mathbf{B}(x)\times v + \varepsilon\,\mathrm{N}(v,\mathbf{E},\mathbf{B})\,, & v(0) = v_0\,. \end{array} \right. \tag{3.1}$$

There remains a large factor, namely ε^{-1}. A *work of preparation* based on Section 2 is needed to see how this singular term acts on the phase space. This results at the end of Subsection 3.1 in a two-scale model, from which we extract in Subsection 3.2 a notion of *mean flow*. This mean flow is periodic. Here lies a key to derive uniform estimates, to eliminate problems of secular growth. This property is the gateway to the *three-scale analysis* that is initiated in Subsection 3.3.

3.1. **A work of preparation.** For the sake of completeness, in Subsection 3.1.1, the origin of the dimensionless formulation (1.1) is clarified. In Subsection 3.1.2, a few comments are added to explain how the study of (3.1) is related to questions arising in fluid dynamics. In Subsection 3.1.3, to provide a better fit with the motions of charged particles, the space coordinate x is changed into (F,ϕ). On the other hand, the velocity v is decomposed in the orthonormal basis $(e^m_{f1},e^m_{f2},e^m_{f3})(\mathrm{F},\phi)$ which is generated at the position (F,ϕ) by the magnetic field. As explained in Subsection 3.1.4, interpreted in this frame field, the fast scale of (3.1) acts on v in a simplified manner, through a group of fast rotations.

3.1.1. *Dimensionless equations.* As mentioned in the introduction, we will only consider the case of electrons (sign $-$). The other case (ions) would require to change the data accordingly. In standard units, the time, space and velocity variables $(\tilde{\mathrm{t}},\tilde{\mathrm{x}},\tilde{\mathrm{v}})\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$ are expressed respectively in seconds $(\tilde{\mathrm{t}})$, meters $(\tilde{\mathrm{x}})$, and meters per seconds $(\tilde{\mathrm{v}})$. Starting from $(\tilde{\mathrm{t}},\tilde{\mathrm{x}},\tilde{\mathrm{v}})$, various changes of scales are possible [5, 13, 14]. The aim here is to adjust the scalings to catch some turbulent phenomena.

Consider a population of electrons whose density in the phase space is denoted by $\mathrm{f}(\tilde{\mathrm{t}},\tilde{\mathrm{x}},\tilde{\mathrm{v}})$. The effects of collisions will be neglected. It follows that the particles interact mainly by some electromagnetic field $(\tilde{\mathbf{E}},\tilde{\mathbf{B}})(\tilde{\mathrm{t}},\tilde{\mathrm{x}})$ which is created collectively. On the other hand, one must take into account the presence of a strong exterior magnetic field $\mathbf{B}(x)$, given by (2.12). In this context, the relativistic Vlasov equation can be written:

$$\partial_{\tilde{\mathrm{t}}}\mathrm{f}+\tilde{\mathrm{v}}\cdot\nabla_{\tilde{\mathrm{x}}}\mathrm{f}-q_e\,\big[\tilde{\mathbf{E}}(\tilde{\mathrm{t}},\tilde{\mathrm{x}})+\tilde{\mathrm{v}}\times\tilde{\mathbf{B}}(\tilde{\mathrm{t}},\tilde{\mathrm{x}})+\mathrm{b}_0\;\tilde{\mathrm{v}}\times\mathbf{B}(\tilde{\mathrm{x}}/\mathrm{r})\big]\cdot\nabla_{\tilde{\mathrm{p}}}\mathrm{f}=0\,. \tag{3.2}$$

The relativistic framework comes from the possible presence of energetic electrons. The velocity $v:=\tilde{\mathrm{v}}/c$ has a magnitude $|v|$ which is limited by 1, and it is linked to the relativistic momentum $p:=\tilde{\mathrm{p}}/m_e\,c$ through (1.2). Change the time $\tilde{\mathrm{t}}$ into $t:=\tilde{\mathrm{t}}/\mathrm{T}$, with T adjusted in such a way that $\mathrm{T}:=c^{-1}\,\mathrm{r}$. In (3.2), $q_e\simeq 1,6\times10^{-19}\,C$ is the (absolute value of the) charge of the electron; $c\simeq 3\times10^{8}\,ms^{-1}$ is the speed of light; $m_e\simeq 9,1\times10^{-31}\,kg$ is the mass of the electron; in agreement with experimental data, take $\mathrm{r}=3\,m$ and $\mathrm{b}_0=5\,teslas$, so that $\mathrm{T}=10^{-8}\,s$. Now, retain that the non dimensional variables are:

$$t:=\frac{\tilde{\mathrm{t}}}{\mathrm{T}}\,,\qquad x:=\frac{\tilde{\mathrm{x}}}{\mathrm{r}}\,,\qquad v:=\frac{\tilde{\mathrm{v}}}{c}\,,\qquad p:=\frac{\tilde{\mathrm{p}}}{m_e\,c}\,,\qquad \mathrm{f}(\tilde{\mathrm{t}},\tilde{\mathrm{x}},\tilde{\mathrm{v}})=f\Big(\frac{\tilde{\mathrm{t}}}{\mathrm{T}},\frac{\tilde{\mathrm{x}}}{\mathrm{r}},\frac{\tilde{\mathrm{v}}}{c}\Big)\,. \tag{3.3}$$

The quantity $\omega_{ce}:=q_e\,\mathrm{b}_0/m_e$ is known as the cyclotron frequency (or also the electron gyrofrequency). Introduce the cyclotron frequency relative to the reference frequency T^{-1}, which is the small dimensionless number:

$$0<\varepsilon:=\frac{m_e}{q_e\,\mathrm{b}_0}\,\frac{1}{\mathrm{T}}\simeq10^{-4}\ll1\,,\qquad 1\ll\frac{1}{\varepsilon}=\frac{q_e\,\mathrm{b}_0\,\mathrm{T}}{m_e}\simeq10^{4}\,. \tag{3.4}$$

Translate (3.2) in the new variables t, x and v (or p). This yields:

$$\partial_t f+v\cdot\nabla_x f-\frac{1}{\varepsilon}\,\Big[\frac{1}{c\,\mathrm{b}_0}\,\tilde{\mathbf{E}}(\mathrm{T}\,t,\mathrm{r}\,x)+\frac{1}{\mathrm{b}_0}\,v\times\tilde{\mathbf{B}}(\mathrm{T}\,t,\mathrm{r}\,x)+v\times\mathbf{B}(x)\Big]\cdot\nabla_p f=0\,. \tag{3.5}$$

Let us make sure that the electromagnetic field $(\tilde{\mathbf{E}},\tilde{\mathbf{B}})(\cdot)$ is indeed a perturbation. In the particular case under study, this amounts to introduce $(\mathbf{E},\mathbf{B})(t,x,v)$ with:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} (\tilde{\mathrm{t}},\tilde{\mathrm{x}},\tilde{\mathrm{v}}) := \begin{pmatrix} c\ \mathrm{b}_0\ \varepsilon^2\, \mathbf{E} \\ \mathrm{b}_0\ \varepsilon^2\, \mathbf{B} \end{pmatrix} \Big(\frac{\tilde{\mathrm{t}}}{\mathrm{T}}, \frac{\tilde{\mathrm{x}}}{\mathrm{r}}, \frac{\tilde{\mathrm{v}}}{c} \Big) . \tag{3.6}$$

Then, the change of t into $\tau = \varepsilon\, t$ gives (1.1). Remark that $\tau \simeq 1$ means that $t \simeq \varepsilon^{-1}$, which corresponds to $10^{-4}\, s$. Thus the interval $[\varepsilon, 1]$ interpreted in the time variable τ covers the period $[10^{-8}\, s, 10^{-4}\, s]$. With regards to fusion applications, such a time span is between the electron transit and the sawtooth crash, exactly where turbulence occurs.

3.1.2. *The underlying fluid framework.* The equation:

$$\partial_t v + (v \cdot \nabla_x) v = \varepsilon^{-1}\ \mathbf{B}(x) \times v\, , \qquad v(0,x) = v_0(x)\, , \tag{3.7}$$

describes the motion of a pressureless gas submitted to a strong inhomogeneous magnetic field. It is a simple model arising in magnetohydrodynamics. The characteristics which are associated with (3.7) are obtained by solving the ordinary differential equation (3.1) in the case $\mathrm{c}(\cdot) \equiv 1$ and $\mathrm{N}(\cdot) \equiv 0$. In [19], the authors analyze the interaction between the penalization and the nonlinear transport term of (3.7). They explain how the flow modifies the phase of the oscillation. This is done on a finite time interval, that is with $t \in [0,T]$ for some $T \in \mathbb{R}_+^*$. In comparison with (3.7), the use of a kinetic description through the Vlasov equation (1.1) has many advantages that are briefly enumerated below:

1. The results of [19] cannot be applied over an extended period of times t. Indeed, the crossing of the characteristics induces the blow-up during intermediate times (for $t \simeq 1$) of the smooth solutions u to (3.7). On the contrary, the bicharacteristics issued from (3.1) remain separate and are defined for all times $t \in \mathbb{R}$. Thus, the kinetic context is necessary to avoid the *singularity formation*;
2. When dealing with fluid turbulence, the right hand term of (3.7) is usually replaced by an arbitrary oscillating forcing term [11]. In magnetized plasmas, the agitation is caused by the Lorentz force. Incidently, as will be seen later, the kinetic framework allows to get around *closure problems*;
3. In view of Theorem 1, the profiles ${}^t(X_j, V_j)(\cdot)$ and the phases $\psi_\star(\cdot)$ of (1.6) depend on the selected domain Ω_j. Except perhaps in the libration case (toy model), the long time behavior of $f(\cdot)$ differs according to the choice of Ω_j. Clearly, such a purely *kinetic feature* cannot be seen within a fluid (MHD) model. The assumption of maxwellianity, at the basis of neoclassical theories [17], is here not satisfied.

3.1.3. *Adapted spatial and velocity coordinates.* From now on, we will work with the flux coordinates F. We will follow the trajectory:

$$\mathbb{R} \ni t \longmapsto {}^t(\mathrm{F},\phi)(t) := \Sigma_c^f \circ x(t) = {}^t\big(\psi(t), \chi(t), \phi(t)\big) \in \mathbb{R}^3\, . \tag{3.8}$$

On the other hand, the velocity $v(t)$ is decomposed in the orthonormal basis $(e^m_{cf1}, e^m_{cf2}, e^m_{cf3})$ which is above the position $\mathrm{F}(t)$. In other words, with O^m_{cf} as in (2.23), we consider:

$$w(t) := \mathrm{O}^m_{cf}(\mathrm{F},\phi)(t)\, v(t)\, , \qquad v(t) = \textstyle\sum_{j=1}^3 w^j(t)\ e^m_{cfj}(\mathrm{F},\phi)(t)\, . \tag{3.9}$$

In view of (3.9), the three components w^j of w represent the coordinates of v in the moving frame $(e^m_{c1}, e^m_{c2}, e^m_{c3})$. The decomposition (3.9) allows to easily isolate inside w the two directions, say $w_\parallel$ (or $v_\parallel$) and $w_\perp$ (or $v_\perp$), which are respectively parallel and perpendicular to the magnetic field:

$$w_\perp := w^1 \; e^m_{c1} + w^2 \; e^m_{c2} \,, \qquad w_\parallel := w^3 \; e^m_{c3} \,, \qquad w = w_\perp + w_\parallel \,. \tag{3.10}$$

The triplet $(\textsc{f}, \phi, w)$ is not strictly speaking a position in a physical phase space, since w cannot be interpreted as the time derivative of the position $(\textsc{f}, \phi)$. However, the use of w facilitates the study of the underlying dynamics. Indeed, the kinetic energy is conserved when passing from v to w in (3.9). We have $|w(t)| = |v(t)|$ for all $t \in \mathbb{R}$. Moreover, as long as $t \simeq 1$, the part $w_\parallel$ (or $v_\parallel$) remains essentially the same, while the part $w_\perp$ (or $v_\perp$) undergoes rotations in the plane orthogonal to the direction e^m_{c3}. In short, we will consider the time evolution of $(\textsc{f}, \phi, w)$. To this end, first exploit (2.26) to compute:

$$\partial_t \begin{pmatrix} \textsc{f} \\ \phi \end{pmatrix} = (d_c \Sigma^f_c)\big(x(t)\big) \; v(t) = \mathrm{D}^f_{cf}(\textsc{f}) \; \mathrm{O}^f_{cf}(\textsc{f}) \; {}^t\mathrm{O}^m_{cf}(\textsc{f}) \; w \,. \tag{3.11}$$

Then, apply the definition given to the matrices $\mathrm{O}^*_{\star\diamond}$ as it is explained in Paragraph 2.4, see especially (2.23), and recall (2.25) to extract:

$$\mathrm{O}^f_{cf} \; {}^t\mathrm{O}^m_{cf} = \begin{pmatrix} {}^t e_\psi \\ {}^t e_\chi \\ {}^t e_\phi \end{pmatrix} \big(\, e_\psi \, , \, \cos\omega \, e_\chi - \sin\omega \, e_\phi \, , \, \sin\omega \, e_\chi + \cos\omega \, e_\phi \, \big) = {}^t\mathrm{O}^m_f \,. \tag{3.12}$$

We turn now to the study of $\partial_t w$. Consider the Lie derivative $\mathcal{L}_v$ along the flow of $x(\cdot)$. This derivative is thereby coordinate invariant. It is also the derivative in the velocity direction. More precisely, given a tensor field f, this is just:

$$\mathcal{L}_v f(x) = (v \cdot \nabla_x) f(x) = ({}^t\mathrm{O}^m_c(x) \, w \cdot \nabla_x) f(x) = \textstyle\sum_{j=1}^3 \; w^j \; \big(e^m_{cj}(x) \cdot \nabla_x\big) f(x) \,. \tag{3.13}$$

From (3.9), we can infer that:

$$\partial_t w(t) = \mathrm{O}^m_c\big(x(t)\big) \, \partial_t v(t) + \mathcal{L}_{v(t)} \mathrm{O}^m_c\big(x(t)\big) \, v(t) \,. \tag{3.14}$$

The action of O^m_c allows to straighten out the field lines. We have $\mathrm{c}(|v|) \, \mathrm{O}^m_c \, (\mathbf{B} \times v) = \mathrm{B}_0 \times w$ where the direction of $\mathrm{B}_0(\cdot)$ is now fixed:

$$\mathrm{B}_0(\textsc{f}, w) = \begin{pmatrix} \mathrm{B}^1_0 \\ \mathrm{B}^2_0 \\ \mathrm{B}^3_0 \end{pmatrix} (\textsc{f}, w) = \mathrm{c}(|w|) \; b(\textsc{f}) \; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{3.15}$$

On the other hand, with $\mathrm{N}(\cdot)$ as in (1.4), we find that $\mathrm{O}^m_c \, \mathrm{N}(v, \mathbf{E}, \mathbf{B}) = \mathrm{N}(w, \textsc{e}, \textsc{b})$ with an electromagnetic perturbation transformed according to:

$$\textsc{e}(\textsc{f}) := \mathrm{O}^m_{cf}(\textsc{f}) \; \mathbf{E} \circ \Sigma^c_f(\textsc{f}) \,, \qquad \textsc{b}(\textsc{f}) := \mathrm{O}^m_{cf}(\textsc{f}) \; \mathbf{B} \circ \Sigma^c_f(\textsc{f}) \,. \tag{3.16}$$

The right-hand term of line (3.14) takes into account the distortion that is induced by the change of basis. It has a special structure, and it can be analyzed in terms of $(\textsc{f}, \phi, w)$. Use (2.31) and consider (2.23) to find:

$$\mathcal{L}_{v(t)} \mathrm{O}^m_c\big(x(t)\big) \; v(t) = \mathrm{B}_1\big(\textsc{f}(t), w(t)\big) \times w(t) \,. \tag{3.17}$$

In (3.17), with $v \equiv v(\text{F},\phi,w) = {}^t\mathrm{O}^m_{cf}(\text{F},\phi)\, w$, the function $\mathrm{B}_1(\cdot)$ is given by:

$$\mathrm{B}_1(\text{F},w) = \begin{pmatrix} \mathrm{B}^1_1 \\ \mathrm{B}^2_1 \\ \mathrm{B}^3_1 \end{pmatrix}(\text{F},w) := \begin{pmatrix} [(v\cdot\nabla_x)^t e^m_{c3}] \circ \Sigma^c_f \cdot e^m_{cf2} \\ [(v\cdot\nabla_x)^t e^m_{c1}] \circ \Sigma^c_f \cdot e^m_{cf3} \\ [(v\cdot\nabla_x)^t e^m_{c2}] \circ \Sigma^c_f \cdot e^m_{cf1} \end{pmatrix}. \tag{3.18}$$

The function $\mathrm{B}_1(\cdot)$ of (3.18) does not involve the variable ϕ. It is not inadvertence. Due to the axisymmetric properties, the metric tensor and the Christoffel symbols do not depend on ϕ. In short, we have to deal with:

$$\begin{cases} \partial_t \begin{pmatrix} \text{F} \\ \phi \end{pmatrix} = \mathrm{D}^f_{cf}(\text{F})\ {}^t\mathrm{O}^m_f(\text{F})\ w\,, \\ \partial_t w = \varepsilon^{-1}\,\mathrm{B}(\varepsilon,\text{F},w)\times w + \varepsilon\ \mathrm{N}(w,\text{E},\text{B})\,, \qquad \mathrm{B} := \mathrm{B}_0(\text{F},w) + \varepsilon\ \mathrm{B}_1(\text{F},w)\,. \end{cases} \tag{3.19}$$

We can complete (3.19) with the initial data:

$$(\text{F},\phi)(0) = (\text{F}_0,\phi_0) := \Sigma^f_c(x_0)\,, \qquad \text{F}_0 = {}^t(\psi_0,\chi_0) := \check{\Sigma}^f_c(x_0)\,, \tag{3.20a}$$

$$w(0) = w_0 := O^m_{cf}(\text{F}_0,\phi_0)\ v_0 = \mathrm{w}_0\ {}^t(\cos\varsigma_0\ \cos\upsilon_0\,,\ \cos\varsigma_0\ \sin\upsilon_0\,,\ \sin\varsigma_0\,)\,. \tag{3.20b}$$

Retain that:

$$|w_0| = |v_0| = \mathrm{w}_0\,, \qquad \varsigma_0 = \arctan\big(|w_{0\parallel}|/|w_{0\perp}|\big) = \arctan\big(|v_{0\parallel}|/|v_{0\perp}|\big)\,. \tag{3.21}$$

The Lorentz force is acting only when $w_0 = {}^t(w^1_0,w^2_0,w^3_0)$ is such that $(w^1_0,w^2_0) \neq (0,0)$, that is when $\varsigma_0 \neq \pi/2$ modulo π. This is the most interesting and generic case. With this in mind, we can start with $\varsigma_0 \in]0,\pi[$.

Assumption 3.1 (axisymmetric electromagnetic perturbation)**.** *We suppose that* $\partial_\phi \text{E} \equiv 0$ *and that* $\partial_\phi \text{B} \equiv 0$.

Since the source term in (3.19) does not depend at all on the variable ϕ, we can isolate inside (3.19) the part concerning ${}^t(\text{F},w)$. The component ϕ can be put aside for the moment. Indeed, it can be ultimately recovered from ${}^t(\text{F},w)$.

3.1.4. *The filtering method.* The charged particles rotate rapidly in small circular orbits which are perpendicular to the magnetic field. The velocity w can be expressed in spherical coordinates with radial distance $\mathrm{w} \in \mathbb{R}^*_+$, azimuth angle $\varsigma \in \mathbb{T}$ and polar angle $\upsilon \in \mathbb{T}$, this means that w and ς are *slow* scales, whereas υ is a *fast* scale. This aspect can be taken into account by changing υ into $\varepsilon^{-1}\,\nu$. This sort of *filtering* method leads to:

$$w \equiv w(\mathrm{w},\varsigma,\upsilon) := \mathrm{w}\ {}^t(\cos\varsigma\ \cos\upsilon\,,\ \cos\varsigma\ \sin\upsilon\,,\ \sin\varsigma\,)\,, \qquad \upsilon := \varepsilon^{-1}\,\nu\,. \tag{3.22}$$

The representation (3.22) of w reveals the role of the variable $\upsilon \in \mathbb{T}$. To place more emphasis on the rapid variations, compute:

$$\partial_\nu w = \frac{1}{\varepsilon}\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times w = \frac{1}{\varepsilon}\ \Lambda\ w\,, \qquad \Lambda := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\,{}^t\Lambda\,. \tag{3.23}$$

In summary, we have identified some effective slow variable $z = {}^t(\psi,\chi,\mathrm{w},\varsigma) \in (\mathbb{R}_+\times\mathbb{T})^2$ and a rapid variable $\nu \in \mathbb{R}$ (while $\upsilon \in \mathbb{T}$). At time $t = 0$:

$$z(0) = z_0 := {}^t(\psi_0,\chi_0,\mathrm{w}_0,\varsigma_0)\,, \qquad \nu(0) = \varepsilon\ \upsilon_0\,. \tag{3.24}$$

Convention 3.1. *From now on, calligraphic fonts like $\mathcal{B}$ or $\mathcal{N}$ will be used to designate expressions depending on (z, υ).*

Replace w as indicated in (3.22). Then, the functions $\mathrm{B}^\star_*(\cdot)$ and $\mathrm{N}(\cdot)$ lead to new expressions $\mathcal{B}^\star_*(\cdot)$ and $\mathcal{N}(\cdot)$ depending on (z, υ). With this convention, for $i \in \{1, 2, 3\}$, we have:

$$\mathrm{B}^i = \mathrm{B}^i_0 + \varepsilon\, \mathrm{B}^i_1 = \mathcal{B}^i(\varepsilon, z, \varepsilon^{-1}\, \nu)\,, \qquad \mathcal{B}^i(\varepsilon, z, \upsilon) := \mathcal{B}^i_0(z) + \varepsilon\, \mathcal{B}^i_1(z, \upsilon)\,. \tag{3.25}$$

In view of (3.15), we have:

$$\mathcal{B}^1_0 \equiv 0\,, \qquad \mathcal{B}^2_0 \equiv 0\,, \qquad \mathcal{B}^3_0(z) = c(\mathrm{w})\ b(\psi, \chi)\,. \tag{3.26}$$

On the other hand, we find:

$$\mathcal{B}^1_1(z, \upsilon) = \mathrm{w}\ \big[(\cos\varsigma\ \upsilon * e^m_{cf1} + \sin\varsigma\ e^m_{cf3}) \cdot (\nabla_x {}^t e^m_{c3}) \circ \Sigma^c_f\big] \cdot e^m_{cf2}\,, \tag{3.27a}$$

$$\mathcal{B}^2_1(z, \upsilon) = \mathrm{w}\ \big[(\cos\varsigma\ \upsilon * e^m_{cf1} + \sin\varsigma\ e^m_{cf3}) \cdot (\nabla_x {}^t e^m_{c1}) \circ \Sigma^c_f\big] \cdot e^m_{cf3}\,, \tag{3.27b}$$

$$\mathcal{B}^3_1(z, \upsilon) = \mathrm{w}\ \big[(\cos\varsigma\ \upsilon * e^m_{cf1} + \sin\varsigma\ e^m_{cf3}) \cdot (\nabla_x {}^t e^m_{c2}) \circ \Sigma^c_f\big] \cdot e^m_{cf1}\,. \tag{3.27c}$$

In the same way, we have:

$$\mathcal{N}(z, \upsilon, \mathrm{E}, \mathrm{B}) = {}^t(\mathcal{N}^1, \mathcal{N}^2, \mathcal{N}^3)(z, \upsilon, \mathrm{E}, \mathrm{B}) := \mathrm{N}\big(w(\mathrm{w}, \varsigma, \upsilon), \mathrm{E}, \mathrm{B}\big)\,. \tag{3.28}$$

Lemma 3.1 (interpretation of the dynamical system)**.** *Solving the differential equation (1.3) amounts to the same thing as looking at the solution ${}^t(z, \nu)(\varepsilon, z_0, \upsilon_0; t)$ of:*

$$\partial_t \begin{pmatrix} z \\ \nu \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ a \end{pmatrix}\Big(\varepsilon, z, \frac{\nu}{\varepsilon}\Big)\,, \qquad \begin{pmatrix} z \\ \nu \end{pmatrix}(0) = \begin{pmatrix} z_0 \\ \varepsilon\, \upsilon_0 \end{pmatrix}. \tag{3.29}$$

The nonlinear source term ${}^t(\mathcal{A}, a)(\varepsilon, z, \upsilon)$ is smooth. It is periodic with respect to $\upsilon \in \mathbb{T}$, with a finite asymptotic expansion in powers of ε. More precisely:

$$\begin{aligned} &\mathcal{A}(\varepsilon, \cdot) = \mathcal{A}_0(\cdot) + \varepsilon\, \mathcal{A}_1(\cdot)\,, && \mathcal{A}_\star = {}^t(\mathcal{A}^1_\star, \mathcal{A}^2_\star, \mathcal{A}^3_\star, \mathcal{A}^4_\star)\,, \\ &a(\varepsilon, \cdot) = a_0(\cdot) + \varepsilon\, a_1(\cdot) + \varepsilon^2\, a_2(\cdot)\,. \end{aligned} \tag{3.30}$$

The spatial part of $\mathcal{A}(\cdot)$ is such that $\mathcal{A}^i_1 \equiv 0$ for all $i \in \{1, 2\}$, and:

$$\begin{pmatrix} \mathcal{A}^1_0 \\ \mathcal{A}^2_0 \end{pmatrix}(z, \upsilon) := \ \mathrm{w}\ \begin{pmatrix} \mathrm{D}^f_{cf1} & 0 & 0 \\ 0 & \mathrm{D}^f_{cf2} & 0 \end{pmatrix}\ {}^t\mathrm{O}^m_f\ \begin{pmatrix} \cos\varsigma\ \cos\upsilon \\ \cos\varsigma\ \sin\upsilon \\ \sin\varsigma \end{pmatrix}. \tag{3.31}$$

The velocity part gives rise to $\mathcal{A}^3_0 = 0$. Moreover, with a function $\mathcal{N}(\cdot)$ as in (3.28) and functions $\Gamma^{f*}_{3\star}(\mathrm{F}, \upsilon)$ as in (2.30)-(2.32), we find:

$$\mathcal{A}^4_0 := \mathrm{w}\ (\cos\varsigma\ \Gamma^{f1}_{31} + \sin\varsigma\ \Gamma^{f1}_{33})\,, \tag{3.32a}$$

$$\mathcal{A}^3_1 := \cos\varsigma\ \cos\upsilon\ \mathcal{N}^1 + \cos\varsigma\ \sin\upsilon\ \mathcal{N}^2 + \sin\varsigma\ \mathcal{N}^3, \tag{3.32b}$$

$$\mathcal{A}^4_1 := \mathrm{w}^{-1}\ (-\sin\varsigma\ \cos\upsilon\ \mathcal{N}^1 - \sin\varsigma\ \sin\upsilon\ \mathcal{N}^2 + \cos\varsigma\ \mathcal{N}^3)\,, \tag{3.32c}$$

On the other hand, retain that:

$$a_0 := c(\mathrm{w})\ b\,, \tag{3.33a}$$

$$a_1 := -\,\mathrm{w}\ \ \sin\varsigma\ (\tan\varsigma\ \Gamma^{f2}_{33} + \Gamma^{f2}_{31}) + \mathcal{B}^3_1\,, \tag{3.33b}$$

$$a_2 := \mathrm{w}^{-1}\ (\cos\varsigma)^{-1}\,(-\sin\upsilon\ \mathcal{N}^1 + \cos\upsilon\ \mathcal{N}^2)\,. \tag{3.33c}$$

Proof. To derive (3.31), just consider the upper part of (3.19), and take into account (3.22). To obtain (3.32), look at the bottom part of (3.19) where w is again replaced as in (3.22):

$$\partial_t \mathrm{w} \begin{pmatrix} \cos\varsigma\, \cos\upsilon \\ \cos\varsigma\, \sin\upsilon \\ \sin\varsigma \end{pmatrix} + \mathrm{w}\ \partial_t \varsigma \begin{pmatrix} -\sin\varsigma\, \cos\upsilon \\ -\sin\varsigma\, \sin\upsilon \\ \cos\varsigma \end{pmatrix} + \frac{1}{\varepsilon}\ \mathrm{w}\ \partial_t \nu \begin{pmatrix} -\cos\varsigma\, \sin\upsilon \\ +\cos\varsigma\, \cos\upsilon \\ 0 \end{pmatrix}$$

$$= \frac{1}{\varepsilon}\ \mathrm{w}\ c(\mathrm{w})\ b \begin{pmatrix} -\cos\varsigma\, \sin\upsilon \\ +\cos\varsigma\, \cos\upsilon \\ 0 \end{pmatrix} + \mathrm{w} \begin{pmatrix} +\sin\varsigma\ \mathcal{B}_1^2 - \cos\varsigma\ \sin\upsilon\ \mathcal{B}_1^3 \\ -\sin\varsigma\ \mathcal{B}_1^1 + \cos\varsigma\ \cos\upsilon\ \mathcal{B}_1^3 \\ \cos\varsigma\ \sin\upsilon\ \mathcal{B}_1^1 - \cos\varsigma\ \cos\upsilon\ \mathcal{B}_1^2 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathcal{N}^1 \\ \mathcal{N}^2 \\ \mathcal{N}^3 \end{pmatrix}.$$

The selection of a_0 as in (3.33a) allows to eliminate the terms with ε^{-1} in factor. The other choices inside (3.32) and (3.33) are easily recognized by looking at the above identity. The last equation of (3.29), the one on ν, can be viewed as an *eikonal* equation, whereas ν can be interpreted as a *phase*. □

The Cauchy-Lipschitz theorem guarantees the local existence and uniqueness of a solution to (3.29), defined on a maximal time interval $[0, T(\varepsilon, z_0, \upsilon_0)]$ with $T(\varepsilon, z_0, \upsilon_0) > 0$. The special oscillating structures of the source term $\mathcal{A}(\cdot)$ confer particular properties on ${}^t(z, \nu)$. These are investigated step by step in the next Subsections.

3.2. **The mean flow.** The function $z(\cdot)$ of (3.29) can be decomposed into a mean principal part, say $\mathrm{Z}(z_0;\cdot)$ plus deviations. The expression $\mathrm{Z}(z_0;t) = {}^t(\Psi, \mathrm{X}, \mathrm{W}, \varsigma)(z_0;t)$ does not depend on $\varepsilon \in]0,1]$. The time evolution of $\mathrm{Z}(z_0;\cdot)$ is deduced from (3.29) by averaging the oscillations (in order to eliminate the fast variable $\upsilon \in \mathbb{T}$) and by neglecting the effects of $O(\varepsilon)$-terms. By this way, one comes to:

$$\partial_t \mathrm{Z} = \bar{\mathcal{A}}_0(\mathrm{Z}), \qquad \mathrm{Z}(0) = z_0, \qquad \bar{\mathcal{A}}_0(\mathrm{Z}) := \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}_0(\mathrm{Z}, \upsilon)\ d\upsilon. \tag{3.34}$$

A priori, the life span $\mathrm{T}(z_0)$ associated to (3.34) is finite. To show that $\mathrm{T}(z_0) = +\infty$, a first stage (Paragraph 3.2.1) is to remark that the dynamics of (X, ς) can be studied separately. A second step (Paragraph 3.2.2) is to prove that the related reduced dynamics are hamiltonian, and moreover that they are completely integrable. It follows a partition of the phase space into subsets (Paragraph 3.2.3), with corresponding action angle variables (Paragraph 3.2.4). The flow induced by (3.34) is therefore global, and it acts on these distinct regions in different ways (Paragraph 3.2.5).

3.2.1. *Foliation of the phase space.* With the functions $\omega(\cdot)$ and $\mathrm{D}^f_{cf\star}(\cdot)$ of (2.15) and (2.27), note simply:

$$\mathcal{D}_{f2}(F) := \mathrm{D}^f_{cf2}(F)\ \sin\omega(F) \in \mathbb{R}^*_+, \qquad \mathcal{D}_{f3}(F) := \mathrm{D}^f_{cf3}(F)\ \cos\omega(F) \in \mathbb{R}^*_+. \tag{3.35}$$

When solving (3.34), the components of Z have different roles.

Lemma 3.2. *[structure of the mean flow] The solution $\mathrm{Z}(z_0;\cdot)$ to (3.34) can be put in the form $\mathrm{Z} = {}^t(\psi_0, \mathrm{X}, \mathrm{w}_0, \varsigma)$. The part (X, ς) can be identified through:*

$$\left\{ \begin{aligned} \partial_t \mathrm{X} &= \mathrm{w}_0\ \mathcal{D}_{f2}(\psi_0, \mathrm{X})\ \sin\varsigma, \\ \partial_t \varsigma &= \mathrm{w}_0\ \mathcal{C}_f(\psi_0, \mathrm{X})\ \cos\varsigma, \end{aligned} \right. \qquad \left\{ \begin{aligned} \mathrm{X}(0) &= \chi_0, \\ \varsigma(0) &= \varsigma_0. \end{aligned} \right. \tag{3.36}$$

Proof. First, exploit (2.25) to compute:

$$ {}^t\mathrm{O}_f^m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\omega \\ \cos\omega \end{pmatrix}. $$

Then, with $\mathrm{D}^f_{cfi}(\cdot)$ as in (2.26), use (2.34). By this way, a straightforward calculation indicates that:

$$ \begin{pmatrix} \bar{\mathcal{A}}_0^1 \\ \bar{\mathcal{A}}_0^2 \end{pmatrix} = \mathrm{w}\ \sin\varsigma \ \begin{pmatrix} 0 \\ \mathrm{D}^f_{cf2}\ \sin\omega \end{pmatrix}, \qquad \begin{pmatrix} \bar{\mathcal{A}}_0^3 \\ \bar{\mathcal{A}}_0^4 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{w}\ \mathcal{C}_f\ \cos\varsigma \end{pmatrix}. \tag{3.37} $$

Since $\bar{\mathcal{A}}_0^1 \equiv 0$ and $\bar{\mathcal{A}}_0^3 \equiv 0$, the components Ψ and W remain constant, equal respectively to ψ_0 and w_0. The content of (3.36) is directly issued from (3.20) and (3.37). ☐

In Paragraph 3.2.2, we fix ψ_0 and w_0, and we concentrate on the flow $(\mathrm{X},\varsigma)(\cdot)$ induced by (3.36). The parameter w_0 can be absorbed by changing t into $\mathrm{w}_0\, t$. It does not affect the form of the trajectories. On the contrary, the parameter ψ_0 has such an impact.

3.2.2. *Reduced hamiltonian structure.* In view of (2.6), (2.15) and (2.27b), the function $\mathcal{D}_{f2}(\cdot)$ is positive. This allows to define the *first potential function*:

$$ \mathrm{V}_1(\psi;\mathrm{X}) := -\int_0^{\mathrm{X}} \mathcal{C}_f(\psi,\chi)\ \mathcal{D}_{f2}(\psi,\chi)^{-1}\ d\chi\,, \qquad (\psi,\mathrm{X}) \in \mathbb{R}_+ \times \mathbb{R}\,. \tag{3.38} $$

In contrast to [13], the function $\mathrm{V}_1(\psi;\cdot)$ has the following remarkable property.

Lemma 3.3. *[periodic behavior of the first potential function] For all $\psi \in \mathbb{R}_+$, the potential function $\mathrm{V}_1(\psi;\cdot)$ is periodic of period $2\,\pi$.*

Proof. Fix $\psi \in \mathbb{R}_+$. When dealing with (3.38), it is easier to work with the variable θ instead of χ. Recall (2.8) which gives:

$$ \theta = \check{\Sigma}^t_{f2}(\psi,\chi)\,, \qquad \partial_\chi \theta = \partial_\chi \check{\Sigma}^t_{f2}(\psi,\chi)\,. \tag{3.39} $$

Through $\psi(\cdot)$ and $\chi(\cdot)$, we have access to $\check{\Sigma}^f_t(\cdot)$. Compute $d_{r,\theta}\check{\Sigma}^f_t(\cdot)$ and the corresponding inverse matrix to extract:

$$ 1 = (\partial_r \psi)^{-1}\,(\partial_\theta \chi\,\partial_r \psi - \partial_r \chi\,\partial_\theta \psi)\ \partial_\chi \check{\Sigma}^t_{f2}\,. \tag{3.40} $$

It follows first from (2.27b) and (3.40), secondly from an interpretation of (2.6b), and finally from (2.16) together with (2.14) that:

$$ \partial_\chi \check{\Sigma}^t_{f2}\ \mathrm{D}^f_{cf2}\ \sin\omega = \frac{\partial_r \psi\ |\nabla_p \chi|\ \sin\omega}{\partial_\theta \chi\,\partial_r \psi - \partial_r \chi\,\partial_\theta \psi} = \frac{\partial_r \psi\ \sin\omega}{r\ |\nabla_p \psi|} = \frac{\partial_r \psi}{r\ a}\,. \tag{3.41} $$

Change the variable χ into θ in the integrand of (3.38), use the above identity (3.41), and apply Lemma 2.2 with $F(a) = a$. This leads to:

$$ \int_0^{2\,\pi} \frac{\mathcal{C}_f}{\mathcal{D}_{f2}}(\psi_0,\chi)\ d\chi = \int_0^{2\,\pi} \Big[\,\frac{r\ \mathcal{C}_t\ a}{\partial_r \psi}\,\Big](\gamma_\psi(\theta))\ d\theta = 0\,. \tag{3.42} $$

The function $(\mathcal{C}_f/\mathcal{D}_{f2})(\psi_0,\cdot)$ is clearly periodic with period $2\,\pi$. Combined with (3.42), this remark gives access to Lemma 3.3. ☐

Introduce also the *second potential function*:

$$\mathrm{V}_2(\varsigma) := -\ln(\cos\varsigma)\,, \qquad \varsigma \in \,]-\pi/2,\pi/2[\,. \tag{3.43}$$

The function $\mathrm{V}_2(\cdot)$ is positive with a strict global minimum at $\varsigma = 0$. With $\mathrm{V}_1(\cdot)$ and $\mathrm{V}_2(\cdot)$, we can construct some auxiliary function $\mathrm{H}(\psi;\cdot)$.

Definition 3.1. *[energy function] The sum of the two potential functions gives rise to:*

$$\mathrm{H}(\psi;\mathrm{X},\varsigma) := \mathrm{V}_1(\psi;\mathrm{X}) + \mathrm{V}_2(\varsigma)\,, \qquad (\psi,\mathrm{X},\varsigma) \in \mathbb{R}_+ \times \mathbb{T} \times]-\pi/2,\pi/2[\,. \tag{3.44}$$

The reason for introducing $\mathrm{H}(\psi;\cdot)$ is clear from what follows.

Lemma 3.4. *[a constant of motion] For all $\psi_0 \in \mathbb{R}_+$ and $\mathrm{w}_0 \in \mathbb{R}_+$, the function $\mathrm{H}(\psi_0,\cdot)$ is constant along the trajectories induced by (3.36), meaning that:*

$$\mathrm{H}\big(\psi_0;\mathrm{X}(z_0;t),\varsigma(z_0;t)\big) = \textsc{h}_0\,, \qquad \forall\, t \in \mathbb{R}\,, \qquad \textsc{h}_0 := \mathrm{H}(\psi_0,\chi_0,\varsigma_0)\,. \tag{3.45}$$

Proof. In view of (3.36), this result is straightforward. □

As a consequence of Lemma 3.3, the function $\mathrm{V}_1(\psi;\cdot)$ is bounded. On the other hand, the function $\mathrm{V}_2(\cdot)$ is positive, and it goes to $+\infty$ when $\varsigma \in]-\pi/2,\pi/2[$ tends to $\pm\pi/2$. Now, let us recall that ς_0 was chosen in the interval $]-\pi/2,\pi/2[$. Look at (3.45). This means that $\varsigma(z_0;t)$ remains confined in a compact set within $]-\pi/2,\pi/2[$ for all $t \in \mathbb{R}$. With this in mind, Lemma 3.4 says that the positions occupied by $(\mathrm{X},\varsigma)(z_0;\cdot)$ stay on the level set of $\mathrm{H}(\psi_0,\cdot)$ with energy $\textsc{h}_0$, that is on $\mathrm{L}(\psi_0,\textsc{h}_0)$ with:

$$\mathrm{L}(\psi_0,\textsc{h}_0) := \big\{\,(\mathrm{X},\varsigma) \in \mathbb{T}\times]-\pi/2,\pi/2[\,;\ \mathrm{H}(\psi_0,\mathrm{X},\varsigma) = \textsc{h}_0\,\big\}\,. \tag{3.46}$$

The two-dimensional system (3.36) being conservative, it can be reduced to quadratures.

Proposition 3.1. *[the reduced hamiltonian dynamics] The system (3.36) is endowed with some hamiltonian structure, associated with the reduced hamiltonian:*

$$\mathcal{H}(\psi_0,q,p) := \mathrm{V}_1(\psi_0,q) + \frac{p^2}{2}\,, \qquad (q,p) \in \mathfrak{R} := \mathbb{T}\times\mathbb{R}\,. \tag{3.47}$$

It is therefore of pendulum type, equivalent to:

$$\left\{\begin{aligned} \partial_{\mathrm{t}} q &= +\,\partial_p \mathcal{H}(\psi_0,q,p) = p\,,\\ \partial_{\mathrm{t}} p &= -\,\partial_q \mathcal{H}(\psi_0,q,p) = -\,\partial_q \mathrm{V}_1(\psi_0,q)\,, \end{aligned}\right. \qquad \left\{\begin{aligned} q(0) &= q_0\,,\\ p(0) &= p_0\,. \end{aligned}\right. \tag{3.48}$$

Proof. The main task is to define t, q and p. Take $\mho_1^1(\mathrm{X}) := \mathrm{X}$ for all $\mathrm{X} \in \mathbb{T}$, as well as:

$$\mho_2^1(0) := 0\,, \quad \mho_2^1(\varsigma) := \sqrt{2}\ \operatorname{sgn}\varsigma\ (-\ln\cos\varsigma)^{1/2}\,, \quad \forall\,\varsigma \in \,]-\pi/2,0[\cup]0,\pi/2[\,. \tag{3.49}$$

The function $\mho_2^1 :]-\pi/2,\pi/2[\longrightarrow \mathbb{R}$ is strictly increasing, with $\mho_2^{1\prime}(0) = 1$. It is bijective and of class $\mathcal{C}^1(\mathbb{R})$. Thus, the following application is a diffeomorphism:

$$\begin{array}{rcl} \mho^1 \ :\ \mathbb{T}\times\,]-\pi/2,\pi/2[& \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (\mathrm{X},\varsigma) & \longmapsto & (q,p) := {}^t\big(\mho_1^1(\mathrm{X}),\mho_2^1(\varsigma)\big)\,. \end{array} \tag{3.50}$$

Given z_0 and some function $f :]-\pi/2,\pi/2[\longrightarrow \mathbb{R}_+^*$, change the time variable t into t with:

$$\mathrm{t} \equiv \mathrm{t}(z_0;t) := \mathrm{w}_0 \int_0^t \mathcal{D}_{f2}\big(\psi_0,\mathrm{X}(z_0;s)\big)\ f\circ\varsigma(z_0;s)\ ds\,. \tag{3.51}$$

In order to recover the hamiltonian formulation (3.48), the functions $f(\cdot)$ and $\mho_2^1(\cdot)$ must be adjusted in such a way that:

$$\sin\varsigma = f(\varsigma)\ \mho_2^1(\varsigma)\,, \qquad f(\varsigma) = \mho_2^{1\prime}(\varsigma)\ \cos\varsigma\,. \tag{3.52}$$

We can eliminate $f(\cdot)$ by combining the above relations to extract $(\mho_2^1\,\mho_2^{1\prime})(\varsigma) = \sin\varsigma/\cos\varsigma$. After integration, this yields $p = \mho_2^1(\varsigma)$ as indicated in (3.49) and (3.50). Then, this gives access to $f(\cdot)$ through the right part of (3.52). As expected, the function $f(\cdot)$ is positive, with $f(0) = 1$. Now, it suffices to take $q_0 := \chi_0$ and $p_0 := \mho_2^1(\varsigma_0)$. □

The solution to (3.48) will be denoted by $(q,p)(\psi_0, q_0, p_0; \mathrm{t})$.

3.2.3. *Phase portrait with respect to* (q,p). According to Sard's lemma, the set:

$$\mathcal{F}(\psi) := \{\, \varkappa \in \mathbb{R}\,;\ \exists\, q \in \mathbb{R} \text{ with } \mathrm{V}_1(\psi,q) = \varkappa \text{ and } \mathcal{C}_f(\psi,q) = 0\,\} \tag{3.53}$$

of stationary values of $\mathrm{V}_1(\psi,\cdot)$ is a bounded closed set of measure zero. To simplify the discussion, we can work under the following hypothesis.

Assumption 3.2. *[finite number of critical values] The set $\mathcal{F}(\psi)$ is finite:*

$$\mathcal{F}(\psi) := \{\, \varkappa_1(\psi), \cdots, \varkappa_m(\psi)\,\}\,, \qquad \varkappa_1(\psi) < \cdots < \varkappa_m(\psi)\,. \tag{3.54}$$

In view of Lemma 2.2, the set $\mathcal{F}(\psi)$ is not empty. The cardinal of $\mathcal{F}(\psi)$ is $m \in \mathbb{N}$ and the upper bound of $\mathcal{F}(\psi)$ is:

$$\varkappa_m(\psi) := \sup\,\{\,\varkappa\,;\ \varkappa \in \mathcal{F}(\psi)\} = \max\,\{\,\varkappa\,;\ \varkappa \in \mathcal{F}(\psi)\}\,. \tag{3.55}$$

Given $\varkappa_j \in \mathcal{F}(\psi)$ and q as in (3.53), the position $(q,0) \in \mathfrak{R}$ is a fixed point of (3.48), which is contained in the level set:

$$\mathscr{L}_j(\psi) := \{\,(q,p) \in \mathfrak{R}\,;\ \mathcal{H}(\psi_0,q,p) = \varkappa_j\,\}\,, \qquad \mathscr{L}(\psi) := \bigcup_{j=1}^{m} \mathscr{L}_j(\psi)\,. \tag{3.56}$$

The separatrices $\mathscr{L}_j(\psi)$ mark boundaries (made of homoclinic orbits) between trajectories corresponding to distinct homotopy classes. Adopt the two conventions $\varkappa_0(\psi) := -\infty$ and $\varkappa_{m+1}(\psi) := +\infty$. Introduce the open domains:

$$\mathfrak{R}L_j(\psi) := \{\,(q,p)\,;\ \varkappa_j(\psi) < \mathcal{H}(\psi,q,p) < \varkappa_{j+1}(\psi)\}\,, \qquad \forall\, j \in \{0,\cdots,m\}\,, \tag{3.57a}$$

$$\begin{aligned}\mathfrak{R}L(\psi) &:= \{\,(q,p)\,;\ \mathcal{H}(\psi,q,p) \notin \mathcal{F}(\psi)\,,\ \mathcal{H}(\psi,q,p) < \varkappa_m(\psi)\}\\ &= \mathfrak{R}L_0(\psi) \cup \cdots \cup \mathfrak{R}L_{m-1}(\psi)\,,\end{aligned} \tag{3.57b}$$

$$\mathfrak{R}R(\psi) := \{\,(q,p)\,;\ \varkappa_m(\psi) < \mathcal{H}(\psi,q,p)\} \equiv \mathfrak{R}L_m(\psi)\,. \tag{3.57c}$$

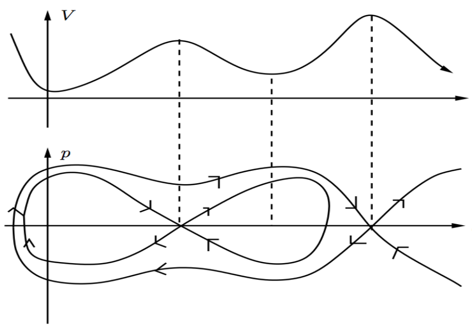
Obviously $\mathfrak{R}L_0(\psi) = \emptyset$. On the contrary, since $\mathcal{H}(\psi,q,\cdot)$ goes to $+\infty$ when $|p| \to +\infty$, the set $\mathfrak{R}R(\psi)$ is not empty. The reduced phase space can be partitioned into:

$$\mathfrak{R} = \mathfrak{R}L(\psi) \cup \mathscr{L}(\psi) \cup \mathfrak{R}R(\psi)\,, \qquad \mathfrak{R}R(\psi) \neq \emptyset\,. \tag{3.58}$$

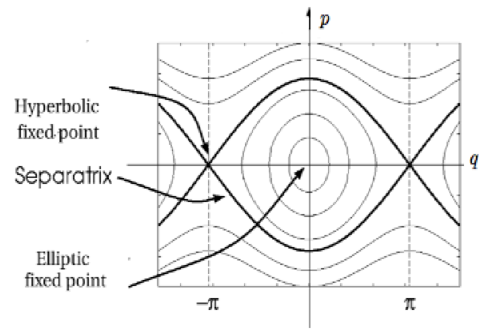
For $j \in \{0,\cdots,m\}$, the open set Ω_j of Theorem 1 is the pullback of $\mathfrak{R}L_j(\psi)$ in the original phase space variables (x,v), that is:

$$\begin{aligned}\Omega_j := \{\,(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3\,;\ &\Sigma_c^f(x) = (\psi,\chi,\phi)\,,\ (q,p) = \mho^1(\chi,\varsigma) \in \mathfrak{R}L_j(\psi)\,,\\ &0 < |v| < 1\,,\ O_{cf}^m(\psi,\chi,\phi)\,v = \mathrm{w}\ (\cos\varsigma\,\cos\upsilon\,,\ \cos\varsigma\,\sin\upsilon\,,\ \sin\varsigma\,)\}\,.\end{aligned} \tag{3.59}$$

Now, two main types of motions may be distinguished:



(a) • Case L • *Libration*. This is inside $\Re L(\psi)$. The *trapped* particles are bouncing back and forth between two poloidal angles ("banana" orbits). They only explore part of the torus.



(b) • Case R • *Rotation*. This is inside $\Re R(\psi)$. The *passing* particles have a net parallel velocity. They are in transit in q with momentum p not changing sign. They explore all parts of the torus.

FIGURE 4. The two main types of motions.

In case L, that is when $(q_0,p_0) \in \Re L(\psi_0)$, let q_1 and q_2 (with $q_1 \leq q_2$) be the turning points associated with the energy level H_0. The solution of (3.48) is periodic with period:

$$\mathcal{P}_L \equiv \mathcal{P}_L(\psi_0,\mathcal{H}_0) := \int_{q_1}^{q_2} \frac{\sqrt{2}}{\sqrt{\mathcal{H}_0 - \mathrm{V}_1(\psi_0,q)}}\, dq\,, \qquad \mathrm{H}_0 \equiv \mathcal{H}_0 := \mathcal{H}(\psi_0,q_0,p_0)\,. \tag{3.60}$$

In case R, that is when $(q_0,p_0) \in \Re R(\psi_0)$, the function $p(\cdot;\psi_0,q_0,p_0)$ is periodic of period:

$$\mathcal{P}_R \equiv \mathcal{P}_R(\psi_0,\mathcal{H}_0) := \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{1}{\sqrt{\mathcal{H}_0 - \mathrm{V}_1(\psi_0,q)}}\, dq\,. \tag{3.61}$$

The other component $q(\cdot)$ satisfies:

$$q(\mathrm{t}) = (2\pi/\mathcal{P}_R)\ \mathrm{t} + q^*(\mathrm{t})\,, \qquad q^*(\mathrm{t}+\mathcal{P}_R) = q^*(\mathrm{t})\,, \qquad \forall\, \mathrm{t} \in \mathbb{R}\,. \tag{3.62}$$

Due to the geometrical identification of q with $q + 2\pi$, the function $q(\psi_0,q_0,p_0;\cdot)$ is also periodic of period $\mathcal{P}_R$. When the potential $\mathrm{V}_1(\psi,\cdot)$ is simply the cosine function, analytic representations (for the period, the action-angle coordinates, the canonical transformations, etc.) are available in terms of Jacobi elliptic functions and complete elliptic integrals [8]. This is what allows in [7] to find compact formulas for the guiding-center orbits. On the contrary, for general functions $\mathrm{V}_1(\psi,\cdot)$, explicit formulas are not available. Only qualitative properties remain. The function $\mathcal{P}_L(\psi,\cdot)$ of (3.60) is increasing on $\Re L(\psi)$, with $\mathcal{P}_L(\psi,\cdot)$ going to $+\infty$ when $\mathcal{H}$ tends from the left to elements in $\mathcal{F}(\psi)$. On the contrary, the function $\mathcal{P}_R(\psi,\cdot)$ as defined in (3.61) is clearly decreasing on $\Re R(\psi)$.

Example 5. *[toy model] In the case of Example 1, we simply find that* $\mathrm{V}_1 \equiv 0$. *The set of critical values is* $\mathcal{F}(\psi) = \{0\}$, *so that* $m = 1$ *and* $\mathcal{H}_1(\psi) = 0$. *Since the function* $\mathrm{V}_2(\cdot)$ *is non negative, we have* $\Re L(\psi) = \emptyset$. *There remains:*

$$\mathscr{L}_1(\psi) \equiv \mathscr{L}(\psi) = \big\{(q,0)\,;\, q \in \mathbb{T}\,\big\}\,, \qquad \Re R(\psi) = \big\{(q,p)\,;\, q \in \mathbb{T}\,,\, p \neq 0\,\big\}\,. \tag{3.63}$$

The motion is just a translation in q with a variable speed (depending on p_0):

$$(q,p)(\psi_0,q_0,p_0;\mathrm{t}) = (q_0 + p_0\,\mathrm{t}, p_0)\,, \qquad \forall\, \mathrm{t} \in \mathbb{R}\,. \tag{3.64}$$

Change the variable t *into $s = p_0\,\mathrm{t}$. The speed becomes constant, equal to 1. Viewed as a function of s, the flow $(q,p)(\cdot)$ is periodic of period 2π. All particles spiral around the field lines, and all gyrocenters make both toroidal and poloidal full rotations. The toy model is concerned only with rotations (R). The periodicity in q is due to the axisymmetric context (no mirror points). This situation is clearly complementary to the one of* [13].

3.2.4. *Action angle variables, consequences.* Fix some $j \in \{0,\cdots,m\}$. Inside $\Re L_j(\psi)$, we can find canonical coordinates, say:

$$(\mathrm{I},\Theta) = \mho^2(\psi,q,p) = {}^t(\mho^2_1,\mho^2_2)(\psi,q,p) = \mho(\psi,\chi,\varsigma) := \mho^2(\psi,\mho^1(\chi,\varsigma))\,. \tag{3.65}$$

The new hamiltonian becomes:

$$H(\psi,\mathrm{I},\Theta) \equiv H(\psi,\mathrm{I}) := \mathcal{H}(\psi,\mho^2(\psi,\cdot)^{-1}(\mathrm{I},\Theta))\,, \qquad (\mathrm{I},\Theta) \in \mathbb{R}\times\mathbb{T}\,. \tag{3.66}$$

We have $\partial_\Theta H \equiv 0$ and:

$$0 < \partial_{\mathrm{I}} H(\psi,\mathrm{I}) = \frac{2\pi}{\mathcal{P}(\psi,\mathcal{H})}\,, \qquad \mathcal{P} := \begin{cases} \mathcal{P}_L(\psi,\mathcal{H}) & \text{if} \quad (q,p)\in \Re L(\psi)\,, \\ \mathcal{P}_R(\psi,\mathcal{H}) & \text{if} \quad (q,p)\in \Re R(\psi)\,. \end{cases} \tag{3.67}$$

Use in place of $(q,p) \in \Re$ the new canonical coordinates (I,Θ) of (3.65) where $\psi = \psi_0$. The system (3.48) is reduced to:

$$\begin{cases} \partial_{\mathrm{t}} \mathrm{I} \;= 0\,, \\ \partial_{\mathrm{t}} \Theta \;= \partial_{\mathrm{I}} H(\psi_0,\mathrm{I})\,, \end{cases} \qquad \begin{cases} \mathrm{I}(0) \;= \mathrm{I}_0 := \mho^2_1(\psi_0,q_0,p_0)\,, \\ \Theta(0) \;= \Theta_0 := \mho^2_2(\psi_0,q_0,p_0)\,. \end{cases} \tag{3.68}$$

The solution to this Cauchy problem (3.68) is globally defined. It is simply:

$$(\mathrm{I},\Theta)(\mathrm{I}_0,\Theta_0;\mathrm{t}) = \big(\,\mathrm{I}_0\,,\,\Theta_0 + \partial_{\mathrm{I}} H(\psi_0,\mathrm{I}_0)\,\mathrm{t}\,\big)\,, \qquad \mathrm{t}\in\mathbb{R}\,. \tag{3.69}$$

Change the time variable t into s with:

$$s \equiv \mathrm{s}(z_0;t) := \partial_{\mathrm{I}} H(\psi_0,\mathrm{I}_0)\ \mathrm{t}(z_0;t)\,, \qquad t\in\mathbb{R}\,. \tag{3.70}$$

Expressed in terms of s, the flow associated to (3.68) becomes a uniform translation:

$$(\mathrm{I},\Theta)(\mathrm{I}_0,\Theta_0;s) = (\,\mathrm{I}_0\,,\,\Theta_0 + s\,)\,, \qquad s\in\mathbb{R}\,. \tag{3.71}$$

It must be clear that the action angle variables (I,Θ) as well as the transformations $\mho^2(\cdot)$ and $\mathrm{s}(\cdot)$ depend on the choice of $j \in \{0,\cdots,m\}$. Now, the mean flow $\mathrm{Z}(z_0;t)$ can also be viewed as a function of s. This amounts to consider the application:

$$\mathrm{Z}_0(z_0;s) = {}^t\big(\psi_0,\mathrm{X}_0(z_0;s),\mathrm{w}_0,\varsigma_0(z_0;s)\big)\,, \qquad s\in\mathbb{R}\,, \tag{3.72}$$

where $\mathrm{Z}_0(\cdot)$ is uniquely determined by the relation:

$$\mathrm{Z}_0\big(z_0;\mathrm{s}(z_0;t)\big) = {}^t\big(\psi_0,\mathrm{X}(z_0;t),\mathrm{w}_0,\varsigma(z_0;t)\big)\,, \qquad \forall\, t\in\mathbb{R}\,. \tag{3.73}$$

With $\star \in \{i,e\}$, consider smooth functions $\tilde{\mathcal{H}}^{+\star}_j : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\tilde{\mathcal{H}}^{-\star}_{j+1} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying:

$$\mathcal{H}_j(\psi) < \mathcal{H}^{+e}_j(\psi) < \mathcal{H}^{+i}_j(\psi) < \mathcal{H}^{-i}_j(\psi) < \mathcal{H}^{-e}_{j+1}(\psi) < \mathcal{H}_{j+1}(\psi)\,, \qquad \forall\,\psi\in\mathbb{R}\,. \tag{3.74}$$

In (3.74), the symbols "i" and "e" stand respectively for $\underline{i}$nterior and $\underline{e}$xterior. In practice the value of ψ is bounded, say by $\psi_m \in \mathbb{R}_+^*$. On the other hand, the velocity can be kept away from the two extreme values $v = 0$ and $|v| = 1$, say $\mathrm{w}_i \leq \mathrm{w} \leq \mathrm{w}_{i+1}$ for some $(\mathrm{w}_i, \mathrm{w}_{i+1}) \in]0,1[^2$. Taking this into account, we can distinguish the compact set:

$$\begin{aligned} K^\star := \{ \, z = {}^t(\psi, \chi, \mathrm{w}, \varsigma) \in (\mathbb{R}_+ \times \mathbb{T})^2 \, ; \, 0 \leq \psi \leq \psi_m \, , \mathrm{w}_i \leq \mathrm{w} \leq \mathrm{w}_{i+1} \, , \\ \mathcal{H}_j(\psi) < \mathcal{H}_j^{+\star}(\psi) \leq \mathrm{H}(\psi; \chi, \varsigma) \leq \mathcal{H}_{j+1}^{-\star}(\psi) < \mathcal{H}_{j+1}(\psi) \, \} \, . \end{aligned} \tag{3.75}$$

As indicated in Theorem 1, select any compact set $\mathrm{K} \Subset \Omega_j$. Any such K can be included inside the pull back of some K^i. In other words, with the notations of (3.59) and (3.75), we can find two functions $\mathcal{H}_j^{+i}(\cdot)$ and $\mathcal{H}_{j+1}^{-i}(\cdot)$ such that:

$$\mathrm{K} \subset \{ \, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \, ; \; {}^t(\psi, \chi, \mathrm{w}, \varsigma) \in K^i \, \} \, . \tag{3.76}$$

Thus, with no loss of generality, the discussion may be restricted to K^i. From now on, we will work with $z_0 \in K^i$. Below, we give us some room for manoeuvre in so far as $z \in K^e$.

Lemma 3.5. *[a periodic flow with uniformly bounded derivatives] The function $\mathrm{Z}_0(\cdot)$ is periodic in s with period $2\,\pi$. Whatever K^e as in (3.75) is, it follows that $\mathrm{Z}_0(\cdot)$ has bounded derivatives of all orders with respect to (z_0, s), uniformly in $(z, s) \in K^e \times \mathbb{R}$. For all $s \in \mathbb{R}$, the application $z \longmapsto Z_0(\cdot; s)$ is invertible, and moreover:*

$$\exists \, c(K^e) \in \mathbb{R}_+^* \, ; \qquad c(K^e) \leq |det \; (d_{z_0} \mathrm{Z}_0)(z; s)| \, , \qquad \forall \, (z, s) \in K^e \times \mathbb{R} \, . \tag{3.77}$$

Proof. The change (3.70) is precisely introduced to normalize the period of $\mathrm{Z}(\cdot)$. It follows that the function $\mathrm{Z}_0(z; \cdot)$ is periodic with period $2\,\pi$, independently on the choice of $z \in K^e$. The derivatives of $\mathrm{Z}_0(\cdot)$ can be computed by using the formula:

$$(\mathrm{X}_0, \varsigma_0)(z_0; s) = \mho(\psi_0, \cdot)^{-1}(\mathrm{I}_0, \Theta_0 + s) \, , \qquad (\mathrm{I}_0, \Theta_0) = \mho(\psi_0, \chi_0, \varsigma_0) \, . \tag{3.78}$$

This implies that all derivatives of $\mathrm{Z}_0(\cdot)$ are again periodic in s with period $2\,\pi$, regardless of the choice of $z \in K^e$. On the other hand, the coefficient $det \, d_{z_0} \mathrm{Z}_0(\cdot)$ is nowhere zero since it comes from a flow, or just consider (3.78). Since it is periodic in s, it is bounded below on the compact set $K^e \times \mathbb{T}$, as indicated in (3.77). □

Remark 3.1. *Lemma 3.5, especially the bound for all $s \in \mathbb{R}$, would not be true if the time variable t had not been changed adequately (into s), if for instance the time variable* t *had been chosen. To be convinced of this fact, just remark that:*

$$\partial_{z_0} \big[\, \mathrm{Z}_0\big(z_0; \partial_{\mathrm{I}} H(\psi_0, \mathrm{I}_0) \, \mathrm{t}\big) \big] = \partial_s \mathrm{Z}_0(z_0; s) \; \partial^2_{z_0 \mathrm{I}} H(\psi_0, \mathrm{I}_0) \; \mathrm{t} + \partial_{z_0} \mathrm{Z}_0(z_0; s) \, . \tag{3.79}$$

Observe that the term with t *in factor gives rise to a linear growth with respect to* t*.*

To gain some uniform stability in t with respect to variations in the initial data, it is essential to look for the specific coordinates (I, Θ) and the special time variable s. As a matter of fact, the right strategy is first to locate the action I and secondly to normalize through a change of time variable the speed of propagation in Θ. In the case R of a pure rotation, these manipulations are simplified.

Example 6. *[toy model] Here, there is no need for the transformation* $\mho^2(\cdot)$*. It suffices to take* $(\mathrm{I},\Theta)\equiv(p,q)$*. The changes of time variable (3.51) and (3.70) become:*

$$\mathrm{t}=\mathrm{w}_0\ r^{-1}\,f(\varsigma_0)\ t\,,\qquad s=p_0\ \mathrm{t}\,,\qquad s\equiv \mathrm{s}(z_0;t)=\mathrm{w}_0\ r^{-1}\,f(\varsigma_0)\ p_0\ t\,. \tag{3.80}$$

Recall that $p_0=\mho^1_2(\varsigma_0)$*. Using (3.52), this yields* $f(\varsigma_0)\,p_0=\sin\varsigma_0$*. This is consistent with a direct analysis of (3.36) that would lead to* $s=\mathrm{w}_0\,r^{-1}\,\sin\varsigma_0\ t$*. In accordance with the next definition (3.83), we find also:*

$$P(z_0)=\frac{2\,\pi\,r}{\mathrm{w}_0\,\sin\varsigma_0}\,,\qquad \mathcal{P}(\psi_0,\mathcal{H}_0)\equiv\mathcal{P}_R(\mathcal{H}_0)=\frac{2\,\pi}{p_0}=\frac{2\,\pi}{\sqrt{2\,\mathcal{H}_0}}\,. \tag{3.81}$$

On the other hand:

$$Z_0(z_0,s)={}^t(\psi_0,\chi_0+s,\mathrm{w}_0,\varsigma_0)\,,\qquad (d_{z_0}\mathrm{Z}_0)(z_0;s)=Id\,,\qquad \forall\,s\in\mathbb{R}\,. \tag{3.82}$$

3.2.5. *Different long-time behaviors.* Let $P(z_0)$ be such that:

$$\begin{cases} \mathrm{t}(z_0;P(z_0))=\mathcal{P}_L(\psi_0,\mathcal{H}_0) & \text{if}\quad (q_0,p_0)\in\Re L(\psi_0)\,,\\ \mathrm{t}(z_0;P(z_0))=\mathcal{P}_R(\psi_0,\mathcal{H}_0) & \text{if}\quad (q_0,p_0)\in\Re R(\psi_0)\,. \end{cases} \tag{3.83}$$

As a consequence of the preceding analysis, we can state the following.

Lemma 3.6. *[periodicity of the mean flow] The solution* $\mathrm{Z}(z_0;\cdot)$ *to (3.34) is periodic in time* t *of period* $P(z_0)$*. In particular, it is globally defined, with* $\mathrm{T}(z_0)=+\infty$*.*

Proof. Indeed, the two components Ψ and W remain constant. On the other hand, with t as in (3.51) and in view of Paragraph 3.2.3, we have:

$$\begin{aligned}(\mathrm{X},\Sigma)(z_0;t+P)&=(\mho^1)^{-1}\big((q,p)(\psi_0,q_0,p_0;\mathrm{t}+P)\big)\\ &=(\mho^1)^{-1}\big((q,p)(\psi_0,q_0,p_0;\mathrm{t})\big)=(\mathrm{X},\Sigma)(z_0;t)\,.\end{aligned}$$ □

With the equation (3.19), we can also extract:

$$\partial_t\phi=e_0(z,\frac{\nu}{\varepsilon})\,,\qquad e_0(z,\upsilon):=\mathrm{w}\ (0,0,\mathrm{D}^f_{cf3})\ {}^t\mathrm{O}^m_f\ \begin{pmatrix}\cos\varsigma\ \cos\upsilon\\ \cos\varsigma\ \sin\upsilon\\ \sin\varsigma\end{pmatrix}. \tag{3.84}$$

The function $(\phi,\nu)(\cdot)$ can also be decomposed into a mean principal part $(\Phi,\mathcal{V})(\cdot)$ plus deviations. The functions Φ and $\mathcal{V}$ express the mean motion of the toroidal angle ϕ and of the phase ν. With $\mathcal{D}_{f3}$ as in (3.35), observe that:

$$\bar e_0(z)=\mathrm{w}\ \mathcal{D}_{f3}(\mathrm{F})\ \sin\varsigma\,,\qquad \bar a_0(z)=a_0(z)=c(\mathrm{w})\ b(\mathrm{F})\,. \tag{3.85}$$

Thus, knowing what $(\mathrm{X},\Sigma)(z_0;\cdot)$ is, the two components Φ and $\mathcal{V}$ can be deduced from:

$$\Phi(z_0;t)=\phi_0+\mathrm{w}_0\int_0^t\mathcal{D}_{f3}(\psi_0,\mathrm{X}(s,z_0))\ \sin\Sigma(s,z_0)\ ds\,, \tag{3.86a}$$

$$\mathcal{V}(z_0;t)=c(\mathrm{w}_0)\int_0^t b(\psi_0,\mathrm{X}(s,z_0))\ ds\,. \tag{3.86b}$$

Apart from exceptional cases, we have:

$$\nexists k\in\mathbb{Z}\,;\qquad \Phi(z_0;t+P(z_0))=\Phi(z_0;t)+2\ k\ \pi\,.$$

On the side of ν, there is also no possible geometrical identification of ν with $\nu + \mathrm{c}$ that could be uniform with respect to $\varepsilon \in]0,1]$. In general, neither Φ nor $\mathcal{V}$ are periodic in t. However, their time behaviors fall under the following notion:

Definition 3.2. *[linear plus periodic function] A function* $\mathrm{F}(\cdot)$ *of a single variable* $t \in \mathbb{R}$ *is said linear plus periodic if there exists some* $c \in \mathbb{R}^*$ *and some periodic function* $\mathrm{F}^*(\cdot)$ *of period* $P \in \mathbb{R}_+^*$ *such that:*

$$\mathrm{F}(t) = c\, t + \mathrm{F}^*(t)\,, \qquad \mathrm{F}^*(t+P) = \mathrm{F}^*(t)\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{3.87}$$

Combine (3.67) and (3.83) to get:

$$\mathrm{s}(z_0;t) = \frac{2\,\pi\, t}{P(z_0)} + \mathrm{s}^\star\Big(z_0; \frac{2\,\pi\, t}{P(z_0)}\Big)\,, \qquad \mathrm{s}^\star(z_0; t+2\,\pi) = \mathrm{s}^\star(z_0;t)\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{3.88}$$

Now, expressions like $\flat(z, \varepsilon^{-1}\,\mathcal{V})$ with $\flat \in \{\mathcal{A}, a, e_0\}$ as in (3.29) are obtained by composing a periodic function (in υ) with the form (3.87). The result belongs to another category.

Definition 3.3. *[quasiperiodic function] A function* $\mathrm{Z}(\cdot)$ *of a single variable* $t \in \mathbb{R}$ *is said quasiperiodic of order* $m \in \mathbb{N}$*, with* $2 \leq m$*, if there exists a function* $\mathrm{QZ}(\cdot)$ *of* m *variables, say* $(t_1, \cdots, t_m) \in \mathbb{R}^m$*, periodic in the sense that:*

$$\mathrm{QZ}(t_1, \cdots, t_j + 2\,\pi, \cdots, t_m) = \mathrm{QZ}(t_1, \cdots, t_m)\,, \qquad \forall\, j \in \{1, \cdots, m\}\,, \tag{3.89}$$

and m *positive rationally independent frequencies* $(\omega_1, \cdots, \omega_m) \in (\mathbb{R}_+^*)^m$ *such that:*

$$\mathrm{Z}(t) = \mathrm{QZ}(\omega_1\, t, \cdots, \omega_m\, t)\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{3.90}$$

After integration of $\partial_t \nu$ or $\partial_t \phi$ through (3.29) and (3.84), it is expected (among other things) that some rapidly oscillating quasiperiodic features emerge.

3.3. **Towards a three-scale analysis.** At first approximation, the solution $z(\cdot)$ to (3.29) is supposed to look like the function $\mathrm{Z}(\cdot)$ of (3.34). As a byproduct of Subsection 3.2, the phase space is foliated by two-dimensional surfaces on which the mean flow $\mathrm{Z}(\cdot)$ is completely integrable. By using action angle variables and by changing t into s, the motions reduce to simple rotations. This geometrical rigidity is crucial. It will allow us to implement here a sort of generalized filtering method.

The application $\mathrm{s}(z_0;\cdot)$ has been introduced at the level of (3.70), as a combination of (3.51) and (3.70). It is well defined only for $z_0 = (\psi_0, \chi_0, \mathrm{w}_0, \varsigma_0)$ with $(\chi_0, \varsigma_0) \in \mathfrak{R}L(\psi_0) \cup \mathfrak{R}R(\psi_0)$. It is bijective, with inverse $t(z_0;s)$. Recall that:

$$0 < \partial_t \mathrm{s}(z_0;t) = \partial_{\mathrm{I}} H(\psi_0, \mathrm{I}_0)\ \mathrm{w}_0\ \mathcal{D}_{f2}(\psi_0, \mathrm{X})\ \mho_2^{1'}(\varsigma)\ \cos\varsigma\,. \tag{3.91}$$

Remark that the derivative $\partial_t \mathrm{s}(z_0;\cdot)$ viewed as a function of $s \in \mathbb{R}$ is periodic of period $2\,\pi$. There is a deep reason for introducing s, which has been explained in Remark 3.1. Change further s into $\uptau := \varepsilon\, s$ (with $\uptau \neq \tau$). Now, we can turn to the study of (3.29). Interpreted in terms of $\uptau$, this yields a solution ${}^t(z,\nu)(\varepsilon, z_0, \upsilon_0; \uptau)$ of:

$$\partial_\uptau \begin{pmatrix} z \\ \nu \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathsf{A} \\ \mathsf{a} \end{pmatrix} \Big(\varepsilon, z; \frac{\uptau}{\varepsilon}, \frac{\nu}{\varepsilon}\Big)\,, \qquad \begin{pmatrix} z \\ \nu \end{pmatrix}(0) = \begin{pmatrix} z_0 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \upsilon_0 \end{pmatrix}. \tag{3.92}$$

By construction, we have:

$$\begin{pmatrix} \mathsf{A} \\ \mathsf{a} \end{pmatrix}(z_0 \,|\, \varepsilon, z; s, \upsilon) := \partial_t \mathrm{s}(z_0; t(z_0; s))^{-1} \begin{pmatrix} \mathcal{A} \\ a \end{pmatrix}(\varepsilon, z, \upsilon)\,. \tag{3.93}$$

The dependence on z_0 of the function ${}^t(\mathsf{A}, \mathsf{a})(\cdot)$ is introduced through the factor $(\partial_t \mathrm{s})^{-1}$. If need be, it will be mentioned. But otherwise, when it has no effect, it will be simply omitted and forgotten as is the case for (3.92).

From now on, we will work locally in time, with $\tau \in [0, \mathcal{T}]$ for some $\mathcal{T} \in \mathbb{R}_+^*$. The source term ${}^t(\mathsf{A}, \mathsf{a})(\cdot)$ is periodic of period 2π both in s and υ. In Paragraph 3.3.1, as a preliminary point, some *operations* on such periodic functions are introduced. They will be used throughout the text. Then, the aim is to remove from (3.92) as much singular terms as possible. To this end, we perform a *lifting procedure* presented in Paragraph 3.3.2. This requires to solve some *homological equation* studied in Paragraph 3.3.3. This results in a new system which is still of the form (3.92) but with a partial desingularization. As stated in Proposition 3.2 which is proved in Paragraph 3.3.4, the source term A may be replaced by A with $\mathrm{A} = O(\varepsilon)$.

3.3.1. *Operations on periodic functions.* Let $Z(s, \upsilon)$ be a function depending on $(s, \upsilon) \in \mathbb{T}^2$. Eventually, the function $Z(\cdot)$ may involve other (not listed) parameters (like ε, z, etc.). Given $s \in \mathbb{T}$, the quantity $\bar{Z}(s)$ is the mean value of $Z(s, \cdot)$ with respect to $\upsilon \in \mathbb{T}$. On the other hand, $\langle \bar{Z} \rangle$ is the mean value of $\bar{Z}(s)$ with respect to $s \in \mathbb{T}$. In other words:

$$\bar{Z}(s) := \frac{1}{2\pi} \int_0^{2\pi} Z(s, \upsilon)\, d\upsilon\,, \qquad \langle \bar{Z} \rangle := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} Z(s, \upsilon)\, ds\, d\upsilon\,. \tag{3.94}$$

The oscillating parts of $Z(s, \cdot)$ and $\bar{Z}(\cdot)$ are respectively $Z^*(s, \upsilon)$ and $\bar{Z}^\star(s)$ with:

$$Z^*(s, \upsilon) := Z(s, \upsilon) - \bar{Z}(s)\,, \qquad \bar{Z}^\star(s) := \bar{Z}(s) - \langle \bar{Z} \rangle\,. \tag{3.95}$$

These operations can be interpreted at the level of the Fourier series of $Z(\cdot)$ according to:

$$Z(s, \upsilon) = \overbrace{\underbrace{c_{(0,0)}}_{\langle \bar{Z} \rangle} + \underbrace{\sum_{p \in \mathbb{Z}^*} c_{(p,0)}\, e^{ips}}_{\bar{Z}^\star(s)}}^{\bar{Z}(s)} + \underbrace{\sum_{(p,q) \in \mathbb{Z} \times \mathbb{Z}^*} c_{(p,q)}\, e^{i(ps + q\upsilon)}}_{Z^*(s,\upsilon)}\,. \tag{3.96}$$

There are also two actions ∂_υ^{-1} and ∂_s^{-1} which correspond to the inverse operators of the derivations ∂_υ and ∂_s (we have $\partial_\upsilon^{-1} \partial_\upsilon = Id$ and $\partial_s^{-1} \partial_s = Id$) with values in the set of functions with zero mean. For instance:

$$\partial_s^{-1} \bar{Z}^\star(s) := \int_0^s \bar{Z}(r)\, dr - \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_0^s \bar{Z}(r)\, dr \Big)\, ds\,, \qquad \langle \partial_s^{-1} \bar{Z}^\star \rangle = 0\,.$$

3.3.2. *The lifting procedure.* The procedure described below is an adaptation of the one presented in [13]. In contrast with [13], the discussion is not global in the phase space. The pendulum specificities force us to work locally, inside Ω_j for some $j \in \{0, \cdots, m\}$, according to the partition (3.58). Define:

$$\mathfrak{D} := \{\, z = {}^t(\psi, \chi, \mathrm{w}, \varsigma)\,;\, (\chi, \varsigma) \in \Re L(\psi) \cup \Re R(\psi) \,\}\,. \tag{3.97}$$

Consider a smooth map:

$$
(3.98) \qquad \begin{array}{rcl} \mathcal{C}^\infty \ni \Xi : [0,1] \times \mathfrak{D} \times \mathbb{T}^2 & \longrightarrow & \mathbb{R}^4 \\ (\varepsilon, \mathfrak{z}, s, \upsilon) & \longmapsto & \Xi(\varepsilon, \mathfrak{z}; s, \upsilon) \, . \end{array}
$$

$$
\begin{tikzcd}
& \mathfrak{z}(\tau) \arrow[d, "\Xi"] \\
\tau \arrow[ur] \arrow[r] & z(\tau)
\end{tikzcd}
$$

FIGURE 5. The lift.

Fix $j \in \{0, \cdots, m\}$ and K^i as in (3.75). Recall Lemma 3.1.

Definition 3.4. *[lift] A lift on $K^i \times [0, \mathcal{T}]$ through $\Xi(\cdot)$ of the function $z(\varepsilon, z_0, \upsilon_0; \tau)$ is a function $\mathfrak{z}(\varepsilon, z_0, \upsilon_0; \tau)$ with $\mathfrak{z} = {}^t(\mathfrak{z}^1, \cdots, \mathfrak{z}^4) \in (\mathbb{R}_+ \times \mathbb{T})^2$ such that for all $(\varepsilon, z_0, \upsilon_0, \tau)$ in the domain $[0,1] \times K^i \times \mathbb{T} \times [0, \mathcal{T}]$, we have:*

$$
(3.99) \qquad z(\varepsilon, z_0, \upsilon_0; \tau) = \Xi\Big(\varepsilon, \mathfrak{z}(\varepsilon, z_0, \upsilon_0; \tau); \frac{\tau}{\varepsilon}, \frac{\nu(\varepsilon, z_0, \upsilon_0; \tau)}{\varepsilon}\Big) \, .
$$

The introduction of a well chosen lift allows to remove the singular term $\varepsilon^{-1} \, \mathsf{A}_0$ from (3.92), while conserving the same form as (3.92), and while staying in the framework of periodic functions of period 2π in both variables s and υ. As a matter of fact:

Proposition 3.2. *[desingularization] There exists $\Xi(\cdot)$ as in (3.98) having the form:*

$$
(3.100) \qquad \Xi(\varepsilon, \mathfrak{z}; s, \upsilon) = \bar{\Xi}_0(\mathfrak{z}; s) + \varepsilon \; \Xi_1^*(\mathfrak{z}; s, \upsilon) \, , \qquad \bar{\Xi}_0(\mathfrak{z}; 0) = \mathfrak{z} \, ,
$$

such that for any K^i as in (3.75), there exists $\mathcal{T} \in \mathbb{R}_+^*$ and a lift $\mathfrak{z}(\cdot)$ on $K^i \times [0, \mathcal{T}]$ through $\Xi(\cdot)$ of $z(\cdot)$ such that (1.3), (3.29) and (3.92) are transformed into:

$$
(3.101) \qquad \partial_\tau \begin{pmatrix} \mathfrak{z} \\ \nu \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{a} \end{pmatrix} \Big(\varepsilon, \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\nu}{\varepsilon}\Big) \, , \qquad \begin{pmatrix} \mathfrak{z} \\ \nu \end{pmatrix}(0) = \begin{pmatrix} z_0 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \mathfrak{z}_1 \\ \upsilon_0 \end{pmatrix} .
$$

The source terms $\mathrm{A}(\varepsilon, \mathfrak{z}; s, \upsilon)$ and $\mathrm{a}(\varepsilon, \mathfrak{z}; s, \upsilon)$ are periodic of period 2π in both variables s and υ. They are smooth functions on $[0,1] \times \mathfrak{D} \times \mathbb{T}^2$. Moreover:

$$
(3.102) \qquad \mathrm{A}_0(\mathfrak{z}; s, \upsilon) := \mathrm{A}(0, \mathfrak{z}; s, \upsilon) = 0 \, , \qquad \forall \, (\mathfrak{z}, s, \upsilon) \in \mathfrak{D} \times \mathbb{T}^2 \, .
$$

The component $\mathfrak{z}_1$ in the initial data of (3.101) depends smoothly on ε, z_0 and υ_0. It can be uniquely determined through the relation:

$$
(3.103) \qquad \mathfrak{z}_1(\varepsilon, z_0; \upsilon_0) + \Xi_1^*\big(z_0 + \varepsilon \, \mathfrak{z}_1(\varepsilon, z_0; \upsilon_0); 0, \upsilon_0\big) = 0 \, .
$$

The proof of Proposition 3.2 is postponed to Paragraph 3.3.4. Below, we begin with a preliminary analysis. Assume that (3.101) is satisfied for some source terms $\mathrm{A}(\cdot)$ and $\mathrm{a}(\cdot)$. Suppose also that $\mathfrak{z}(\tau)$ stays for $\tau \in [0, \mathcal{T}]$ with $\mathcal{T} \in \mathbb{R}_+^*$ in a compact set K^e as in (3.74) and (3.75). With $z(\cdot)$ as in (3.99), the condition on $\nu(\cdot)$ in (3.101) coincides with the one on $\nu(\cdot)$ in (3.92) if and only if:

$$
(3.104) \qquad \mathrm{a}(\varepsilon, \mathfrak{z}; s, \upsilon) := \mathsf{a}\big(\varepsilon, \Xi(\varepsilon, \mathfrak{z}; s, \upsilon); s, \upsilon\big) = \mathrm{a}_0 + \varepsilon \; \mathrm{a}_1 + \cdots \, .
$$

Remark that the amplitude of $\mathsf{a}(\cdot)$ cannot be modified when moving from $\mathsf{a}(\cdot)$ to $\mathrm{a}(\cdot)$ as prescribed in (3.104). The equation (3.101) implies the large factor ε^{-1}, at least in factor of $\mathrm{a}(\cdot)$. To be sure that the function $z(\cdot)$ obtained through (3.99) after solving (3.101) is a solution to (3.92), it is necessary to adjust $\Xi(\cdot)$ and $\mathrm{A}(\varepsilon, \cdot) = \mathrm{A}_0(\cdot) + \varepsilon \, \mathrm{A}_1(\cdot) + \cdots$ so that:

$$
(3.105) \qquad (d_{\mathfrak{z}} \Xi \; \mathrm{A} + \partial_s \Xi + \varepsilon^{-1} \, \mathrm{a} \; \partial_\upsilon \Xi)(\varepsilon, \mathfrak{z}; s, \upsilon) = \mathsf{A}\big(\varepsilon, \Xi(\varepsilon, \mathfrak{z}; s, \upsilon); s, \upsilon\big) \, .
$$

The next aim is to investigate (3.105) under the constraint $\mathrm{A}_0 \equiv 0$. Since $\mathsf{a}(\cdot)$ and $\mathsf{A}(\cdot)$ are known functions, this yields conditions on $\Xi(\cdot)$. These conditions are called the homological equation. They are studied separately, in the paragraph below.

3.3.3. *The homological equation.* Take into account (3.100) and (3.102). The leading order term (in powers of ε) issued from (3.105) is just:

$$\partial_s \bar{\Xi}_0 + \mathsf{a}_0(\bar{\Xi}_0; s, \upsilon)\ \partial_\upsilon \Xi_1^* = \mathsf{A}_0(\bar{\Xi}_0; s, \upsilon)\,, \qquad \mathsf{A}_0 = {}^t(\mathsf{A}_0^1, \cdots, \mathsf{A}_0^4)\,. \tag{3.106}$$

In view of (1.4), (2.14) and (3.33a), we have:

$$\partial_\upsilon \mathsf{a}_0 \equiv 0\,, \qquad 0 < \mathsf{a}_0(z; s)\,, \qquad \forall\,(z, s) \in (\mathbb{R}_+ \times \mathbb{T})^2 \times \mathbb{R}\,. \tag{3.107}$$

Take the average of (3.106) with respect to υ. By this way, we can extract the *first modulation equation*:

$$\partial_s \bar{\Xi}_0 = \bar{\mathsf{A}}_0(\bar{\Xi}_0; s)\,, \qquad \bar{\Xi}_0(\mathfrak{z}; 0) = \mathfrak{z}\,, \qquad \bar{\mathsf{A}}_0 = (\partial_t \mathrm{s})^{-1}\, \bar{\mathcal{A}}_0\,. \tag{3.108}$$

The equation (3.108), interpreted in terms of the time variable t, amounts to the same thing as (3.34). With $\mathrm{Z}_0(\cdot)$ as in (3.73), this means that $\bar{\Xi}_0(\cdot) \equiv \mathrm{Z}_0(\cdot)$. Thus, by applying Lemma 3.5, we can state that $\bar{\Xi}_0(\mathfrak{z}; \cdot)$ is globally defined, and that it is periodic in s of period $2\,\pi$. This property is essential. It was one of the goals of Subsection 3.2 to exhibit it. We know what the function $\bar{\Xi}_0(\cdot)$ is. We can therefore come back to (3.106). To obtain (3.106), it suffices now to adjust Ξ_1^* in the following way:

$$\Xi_1^*(\mathfrak{z}; s, \upsilon) = \mathsf{a}_0\big(\bar{\Xi}_0(\mathfrak{z}; s); s\big)^{-1}\ \partial_\upsilon^{-1} \mathsf{A}_0^*\big(\bar{\Xi}_0(\mathfrak{z}; s); s, \upsilon\big) = a_0(\bar{\Xi}_0)^{-1}\ \partial_\upsilon^{-1} \mathcal{A}_0^*(\bar{\Xi}_0; \upsilon)\,. \tag{3.109}$$

Fix any compact set K^e as in (3.74) and (3.75). For $\varepsilon < \varepsilon_0$ with $\varepsilon_0(K^e)$ small enough, the application $\Xi(\cdot)$ is a small perturbation of the flow $\mathrm{Z}_0(\cdot)$. Since $\mathrm{Z}_0(\cdot)$ is periodic in s and since $\mathrm{Z}_0(\cdot; s)$ is invertible for all $s \in \mathbb{T}$ (Lemma 3.5), as a consequence of the implicit function theorem, the application $\Xi(\varepsilon, \cdot; s, \upsilon)$ is also invertible. Moreover, we can find a constant $c(K^e) \in \mathbb{R}_+^*$ such that:

$$0 < c(K^e) < |det\ d_{\mathfrak{z}} \Xi(\varepsilon, \mathfrak{z}; s, \upsilon)|\,, \qquad \forall\,(\varepsilon, \mathfrak{z}, s, \upsilon) \in [0, \varepsilon_0] \times K^e \times \mathbb{T}^2\,. \tag{3.110}$$

The matrix function $(d_{\mathfrak{z}} \Xi)^{-1}(\cdot)$ is therefore well defined, and it has a smooth complete expansion in powers of $\varepsilon \in [0, \varepsilon_0]$, with in particular:

$$d_{\mathfrak{z}} \Xi(\varepsilon, \mathfrak{z}; s, \upsilon)^{-1} = d_{\mathfrak{z}} \bar{\Xi}_0(\mathfrak{z}; s)^{-1} + O(\varepsilon)\,, \qquad \forall\,(\mathfrak{z}, s, \upsilon) \in K^e \times \mathbb{T}^2\,. \tag{3.111}$$

Now, to solve (3.105), it suffices to determine $\mathrm{A}(\cdot)$ by inverting the matrix $d_{\mathfrak{z}} \Xi$. By this way, we find $\mathrm{A} = \varepsilon\, \mathrm{A}_1 + O(\varepsilon)$ with:

$$\begin{aligned} \mathrm{A}_1 := (d_{\mathfrak{z}} \bar{\Xi}_0)^{-1}\ \big\{ & \mathsf{A}_1(\bar{\Xi}_0; s, \upsilon) + (\Xi_1^* \cdot \nabla_{\mathfrak{z}}) \mathsf{A}_0(\bar{\Xi}_0; s, \upsilon) \\ & - \partial_s \Xi_1^* - \mathsf{a}_1(\bar{\Xi}_0; s, \upsilon)\, \partial_\upsilon \Xi_1^* - (\Xi_1^* \cdot \nabla_{\mathfrak{z}}) \mathsf{a}_0(\bar{\Xi}_0; s)\ \partial_\upsilon \Xi_1^* \big\}\,. \end{aligned} \tag{3.112}$$

The choice of $\Xi(\cdot)$ in (3.100) may seem limited. Supplementary terms like $\varepsilon\, \bar{\Xi}_1 + \varepsilon^2\, \Xi_2^* + \cdots$ could be included in order to absorb A_1. As in (3.108), this would yield a differential equation in s on $\bar{\Xi}_1$. However, there is no assurance that the corresponding flow is globally well-defined, preventing apparently such a procedure from being completed.

3.3.4. *Proof of Proposition 3.2.* To match the initial data of (3.92) and (3.101), in view of (3.100), we have to impose:

$$z(0)=z_0=\Xi(\varepsilon,z_0+\varepsilon\,\mathfrak{z}_1;0,\upsilon_0)=z_0+\varepsilon\,\mathfrak{z}_1+\varepsilon\;\Xi_1^*(z_0+\varepsilon\,\mathfrak{z}_1;0,\upsilon_0)\,.$$

This gives access to the relation (3.103). Just apply the Implicit Function Theorem at the level of (3.103) to recover $\mathfrak{z}_1$ with:

$$\mathfrak{z}_1(\varepsilon,z_0;\upsilon_0)=\,-\,\Xi_1^*(z_0;0,\upsilon_0)+O(\varepsilon)\,. \tag{3.113}$$

Select $\mathrm{A}(\cdot)$ and $\mathrm{a}(\cdot)$ as indicated in Paragraph 3.3.3. By construction, we have (3.102), implying that the expression $\varepsilon^{-1}\,\mathrm{A}(\varepsilon,\mathfrak{z};s,\upsilon)$ is bounded locally in $\mathfrak{z}$ and uniformly in ε, s and υ. For $z_0\in K^i\subset K^e$ and for all $\tau\in[0,\mathcal{T}]$ with $\mathcal{T}\in\mathbb{R}_+^*$ small enough, the solution $\mathfrak{z}(\varepsilon,z_0,\upsilon_0;\tau)$ to (3.101) stays in the compact set K^e. Remark that the time $\mathcal{T}$ can depend on the selection of K^e. Define $z(\cdot)$ as in (3.99). As a result of (3.103), (3.104) and (3.105), the function ${}^t(z,\nu)(\cdot)$ is subjected to (3.92) as required.

4. Anomalous transport.

The neoclassical model often fails to provide an accurate description of experimental results. As a matter of fact, it tends to underestimate transport by one order of magnitude or more. The difference between actual transport and the neoclassical expectation is usually called anomalous transport [1, 16]. This discrepancy comes from the oscillations inside (1.6).

Tokamak plasmas are mainly collisionless and in a state not in thermodynamic equilibrium. They involve a large disparity of scales between the gyration time $\tau\simeq\varepsilon^2$ up to the transport time $\tau\simeq 1$. This results in the singular factors ε^{-2} and ε^{-1} of (1.3), associated with fast rotations. As time passes, encoded in the flow of (1.3), the plasmas develop a plethora of mesoscopic oscillations implying complex structures, and giving the impression of a turbulent motion. Still, it is possible to provide a complete deterministic description of this apparently chaotic behavior. This is achieved here by using Wentzel-Kramers-Brillouin methods (WKB approximations) and tools issued from geometric optics [21, 23].

The starting point of this Section 4 is the differential equation (3.101). The aim is to derive a complete description of ${}^t(\mathfrak{z},\nu)(\cdot)$ in terms of an asymptotic series in powers of $\varepsilon\in]0,1]$. More precisely, given $N\in\mathbb{N}$, we seek the solution ${}^t(\mathfrak{z},\nu)(\cdot)$ to (3.101) in the form:

$$\begin{pmatrix}\mathfrak{z}_\varepsilon\\ \nu_\varepsilon\end{pmatrix}(\tau)=\begin{pmatrix}\mathfrak{z}^a_\varepsilon\\ \nu^a_\varepsilon\end{pmatrix}(\tau)+\varepsilon^N\begin{pmatrix}r^{\mathfrak{z}}_\varepsilon\\ r^{\nu}_\varepsilon\end{pmatrix}(\tau),\qquad \begin{pmatrix}\mathfrak{z}^a_\varepsilon\\ \nu^a_\varepsilon\end{pmatrix}(\tau)=\begin{pmatrix}\mathcal{Z}_\varepsilon\\ \mathcal{V}_\varepsilon\end{pmatrix}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\nu_\varepsilon(\tau)}{\varepsilon}\Big)\,. \tag{4.1}$$

The remainders $r^{\mathfrak{z}}_\varepsilon$ and $\varepsilon\,r^{\nu}_\varepsilon$ are intended to satisfy uniform estimates in the sup norm with respect to $\varepsilon\in]0,1]$. On the other hand, the expressions $\mathcal{Z}_\varepsilon$ and $\mathcal{V}_\varepsilon$ are supposed to be given by asymptotic expansions of the following type:

$$\begin{pmatrix}\mathcal{Z}_\varepsilon\\ \mathcal{V}_\varepsilon\end{pmatrix}(\tau,s,\upsilon)=\varepsilon^{-1}\begin{pmatrix}0\\ \langle\bar{\mathcal{V}}_{-1}\rangle\end{pmatrix}(\tau)+\sum_{j=0}^{N-1}\varepsilon^j\begin{pmatrix}\bar{\mathcal{Z}}_j\\ \bar{\mathcal{V}}_j\end{pmatrix}(\tau,s)+\sum_{j=1}^{N}\varepsilon^j\begin{pmatrix}\mathcal{Z}^*_j\\ \mathcal{V}^*_j\end{pmatrix}(\tau,s,\upsilon)\,. \tag{4.2}$$

What comes from (3.29) and (3.101) furnishes:

$$\begin{pmatrix} \mathfrak{Z}_\varepsilon \\ \mathcal{V}_\varepsilon \end{pmatrix} (0,0,\upsilon_0) = \begin{pmatrix} z_0 + \varepsilon\, \mathfrak{z}_1(\varepsilon, z_0; \upsilon_0) \\ \varepsilon\, \upsilon_0 \end{pmatrix}. \tag{4.3}$$

In (4.2), the profiles $\mathfrak{Z}_j(\cdot)$ and $\mathcal{V}_j(\cdot)$ depend smoothly on $(\tau, s, \upsilon) \in \mathbb{R}_+ \times \mathbb{T}^2$. Observe that:

$$\partial_\tau \left[\begin{pmatrix} \mathfrak{Z}_\varepsilon \\ \mathcal{V}_\varepsilon \end{pmatrix} \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu_\varepsilon(\tau)}{\varepsilon}\Big) \right] = \left[Op(\mathfrak{Z}_\varepsilon; \partial) \begin{pmatrix} \mathfrak{Z}_\varepsilon \\ \mathcal{V}_\varepsilon \end{pmatrix} \right] \Big(\tau, \frac{\tau}{\varepsilon}, \frac{\nu_\varepsilon(\tau)}{\varepsilon}\Big), \tag{4.4}$$

where, taking into account (3.101) to replace $\partial_\tau \nu_\varepsilon$, we have introduced:

$$Op(\mathfrak{z}; \partial) \equiv Op(\varepsilon, \mathfrak{z}; s, \upsilon; \partial_\tau, \partial_s, \partial_\upsilon) := \partial_\tau + \varepsilon^{-1} \partial_s + \varepsilon^{-2}\, \mathrm{a}(\varepsilon, \mathfrak{z}; s, \upsilon)\, \partial_\upsilon\,. \tag{4.5}$$

Plug the ansatz (4.1) into (3.101). To obtain formal solutions, we have to impose:

$$Op(\mathfrak{Z}_\varepsilon; \partial) \begin{pmatrix} \mathfrak{Z}_\varepsilon \\ \mathcal{V}_\varepsilon \end{pmatrix} (\tau, s, \upsilon) - \frac{1}{\varepsilon} \begin{pmatrix} \mathrm{A} \\ \mathrm{a} \end{pmatrix} (\varepsilon, \mathfrak{Z}_\varepsilon; s, \upsilon) = \varepsilon^N \begin{pmatrix} R^{\mathfrak{z}} \\ R^{\nu} \end{pmatrix} (\varepsilon; \tau, s, \upsilon) = O(\varepsilon^N). \tag{4.6}$$

The unknown $\nu_\varepsilon(\cdot)$ is no more present at the level of (4.6). This is what the filtering method and the multiscale approach are about. They allow to separate fast from slow scales.

Proposition 4.1. *For all $N \in \mathbb{N}$, the condition (4.6) is satisfied by a formal solution ${}^t(\mathfrak{Z}_\varepsilon, \mathcal{V}_\varepsilon)(\cdot)$ given by (4.2), subjected to the initial condition (4.3). This formal solution is uniquely determined modulo a precision in the sup-norm of the order $O(\varepsilon^{N-1})$.*

Proposition 4.1 is proved in the next subsection. Then, the matter is to recover the exact solutions $\mathfrak{z}_\varepsilon(\cdot)$ and $\nu_\varepsilon(\cdot)$ from the approximate solutions $\mathfrak{Z}_\varepsilon(\cdot)$ and $\mathcal{V}_\varepsilon(\cdot)$, with eventually a fixed lost of precision (in terms of negative powers of ε). This second step is achieved in Subsection 4.2, through Proposition 4.2.

4.1. **Profile equations.** There is another interesting aspect of (4.6). The part involving the component $\mathfrak{Z}_\varepsilon$ does not involve the profile $\mathcal{V}_\varepsilon$. Due to this decoupling property, the formal analysis of (4.6) is primarily concerned with the condition:

$$Op(\mathfrak{Z}_\varepsilon; \partial)\, \mathfrak{Z}_\varepsilon(\tau, s, \upsilon) - \frac{1}{\varepsilon}\, \mathrm{A}(\varepsilon, \mathfrak{Z}_\varepsilon; s, \upsilon) = \sum_{j=-2}^{N-1} \varepsilon^j\, \mathcal{L}_j(\mathfrak{Z}_0, \cdots, \mathfrak{Z}_{j+2}) + O(\varepsilon^N)\,. \tag{4.7}$$

The contribution $\mathcal{L}_j(\cdot)$ is obtained by collecting the terms with the power ε^j in factor, after substitution inside (4.6) of $\mathfrak{Z}_\varepsilon(\cdot)$ by the expansion (4.2). Now, consider the two first terms of the sum in the right hand side of (4.7). Since $\partial_\upsilon \bar{\mathfrak{Z}}_0 \equiv 0$, there is no contribution of size ε^{-2}. Furthermore, since $\mathrm{A}_0 \equiv 0$ as a consequence of (3.102), the restriction $\mathcal{L}_{-1} \equiv 0$ reduces to $\partial_s \bar{\mathfrak{Z}}_0 + \mathrm{a}_0\, \partial_\upsilon \mathfrak{Z}_1^* = 0$. In view of (3.107), this amounts to the same thing as:

$$\bar{\mathfrak{Z}}_0(\tau, s) = \langle \bar{\mathfrak{Z}}_0 \rangle(\tau)\,, \qquad \mathfrak{Z}_1^*(\tau, s, \upsilon) = \bar{\mathfrak{Z}}_1(\tau, s)\,, \qquad \forall\, (\tau, s, \upsilon) \in [0, \mathcal{T}] \times \mathbb{T}^2\,. \tag{4.8}$$

Let us impose (4.8). Then, the sum inside (4.7) starts at $j = 0$. In Paragraph 4.1.1, we study the content of the equation $\mathcal{L}_0 \equiv 0$. In Paragraph 4.1.3, we solve by induction the cascade of equations $\mathcal{L}_j \equiv 0$, for $j = 1$ up to $j = N - 1$. Then, in Paragraph 4.1.4, we explain how to recover from (4.6) the remaining component $\mathcal{V}_\varepsilon$, with $\mathcal{V}_\varepsilon$ as in (4.3).

4.1.1. *A notion of long time gyrokinetic equation.* With A_1 as in (3.112), we find:

$$\mathcal{L}_0(\langle\bar{\mathfrak{Z}}_0\rangle,\bar{\mathfrak{Z}}_1,\mathfrak{Z}_2)=\partial_\tau\langle\bar{\mathfrak{Z}}_0\rangle+\partial_s\bar{\mathfrak{Z}}_1+\mathrm{a}_0(\langle\bar{\mathfrak{Z}}_0\rangle;s)\ \partial_\upsilon\mathfrak{Z}_2^*-\mathrm{A}_1(\langle\bar{\mathfrak{Z}}_0\rangle;s,\upsilon)\ =\ 0\,. \tag{4.9}$$

Take the average of (4.9) in both variables $s\in\mathbb{T}$ and $\upsilon\in\mathbb{T}$ to exhibit what can be viewed as the *second modulation equation*:

$$\partial_\tau\langle\bar{\mathfrak{Z}}_0\rangle\ -\ \langle\bar{\mathrm{A}}_1\rangle(\langle\bar{\mathfrak{Z}}_0\rangle)\ =\ 0\,,\qquad \langle\bar{\mathfrak{Z}}_0\rangle(0)=z_0\ . \tag{4.10}$$

From (4.9), we can also extract:

$$\bar{\mathfrak{Z}}_1^\star=(\partial_s^{-1}\bar{\mathrm{A}}_1^\star)(\langle\bar{\mathfrak{Z}}_0\rangle(\tau);s)\,,\qquad \mathfrak{Z}_2^*=\mathrm{a}_0\big(\langle\bar{\mathfrak{Z}}_0\rangle(\tau);s\big)^{-1}\ \partial_\upsilon^{-1}\mathrm{A}_1^*(\langle\bar{\mathfrak{Z}}_0\rangle(\tau);s,\upsilon)\,. \tag{4.11}$$

As noted in (3.93), the function $\mathrm{A}_1(\cdot)$ depends also on the initial data z_0. More accurately, it should be written $\mathrm{A}_1(z_0\mid\mathfrak{z};s,\upsilon)$. By restricting $\mathcal{T}\in\mathbb{R}_+^*$ if necessary, the nonlinear differential equation (4.10) has a solution $\langle\bar{\mathfrak{Z}}_0\rangle(\cdot)$ on the time interval $[0,\mathcal{T}]$. Then, to solve (4.9), it suffices to adjust $\bar{\mathfrak{Z}}_1^\star(\cdot)$ and $\mathfrak{Z}_2^*(\cdot)$ as indicated in (4.11).

Combine (3.99) and (3.100) together with (4.1), (4.2) and (4.8) to see that:

$$z_\varepsilon(\tau)=\bar{\Xi}_0\Big(\langle\bar{\mathfrak{Z}}_0\rangle(z_0;\tau);\frac{\tau}{\varepsilon}\Big)+O(\varepsilon)\ ,\qquad \forall\,\tau\in[0,\mathcal{T}]\,. \tag{4.12}$$

In view of (4.12), the long time behavior of the function $z_\varepsilon(\cdot)$ is similar to a *large amplitude* oscillation involving the profile $\bar{\Xi}_0(\cdot)$ issued from the mean flow, and a slow modulation governed by $\langle\bar{\mathfrak{Z}}_0\rangle(\cdot)$. Knowing what $\bar{\Xi}_0(\cdot)$ is, the leading-order behavior of the plasma is therefore revealed through $\langle\bar{\mathfrak{Z}}_0\rangle(\cdot)$. From that perspective, we can say that the differential equation (4.10) plays the part of a *long time gyrokinetic equation.*

It may seem a bit disturbing that the equation (4.10) would be formulated in terms of the variable $\mathfrak{z}$ whose physical interpretation is not so readily apparent. On the one hand, this is unavoidable. On the other hand, since $\bar{\Xi}_0\equiv\mathrm{Z}_0$ with Z_0 as in (3.73), the relation (4.12) provides a correspondence between $\langle\bar{\mathfrak{Z}}_0\rangle$ and z_ε, which furnishes in particular:

$$z_\varepsilon^1(\tau)\equiv\psi_\varepsilon(\tau)=\langle\bar{\mathfrak{Z}}_0^1\rangle(z_0;\tau)+O(\varepsilon)\ ,\qquad z_\varepsilon^3(\tau)\equiv\mathrm{w}_\varepsilon(\tau)=\langle\bar{\mathfrak{Z}}_0^3\rangle(z_0;\tau)+O(\varepsilon)\,. \tag{4.13}$$

As a consequence, in contrast with $z_\varepsilon^2\equiv\chi_\varepsilon$ and $z_\varepsilon^4\equiv\varsigma_\varepsilon$, the two components $z_\varepsilon^1\equiv\psi_\varepsilon$ and $z_\varepsilon^4\equiv\mathrm{w}_\varepsilon$ are not oscillating with a large amplitude. At main order, they behave as $\langle\bar{\mathfrak{Z}}_0^1\rangle(z_0;\tau)$ and $\langle\bar{\mathfrak{Z}}_0^3\rangle(z_0;\tau)$, whose physical meaning is therefore clear.

Let us now examine the structure of the source term $\bar{\mathrm{A}}_1(\cdot)$. To this end, supplementary information is needed on the functions $\mathcal{A}_j(\cdot)$ and $a_j(\cdot)$ of Lemma 3.1. In view of the definitions (3.31) and (3.32), with $N_0=2$ and $N_1=3$, for $j=0$ or $j=1$, the Fourier series expansions with respect to the variable $\upsilon\in\mathbb{T}$ of the functions $\mathcal{A}_j(z,\cdot)$ take the form:

$$\mathcal{A}_j(z,\upsilon)=\bar{\mathcal{A}}_j(z)+\sum_{n=1}^{N_j}\mathcal{CA}_{jn}(z)\ \cos(n\,\upsilon)+\sum_{n=1}^{N_j}\mathcal{SA}_{jn}(z)\ \sin(n\,\upsilon)\,. \tag{4.14}$$

In the same way, with $\tilde{N}_1=2$ and $\tilde{N}_2=3$, we have for $j\in\{1,2\}$:

$$a_j(z,\upsilon)=\bar{a}_j(z)+\sum_{n=1}^{\tilde{N}_j}\mathcal{C}a_{jn}(z)\ \cos(n\,\upsilon)+\sum_{n=1}^{\tilde{N}_j}\mathcal{S}a_{jn}(z)\ \sin(n\,\upsilon)\,. \tag{4.15}$$

The nonlinear model (4.10) produces some interaction term $\bar{\mathcal{A}}_1^g(z)$ given by:

$$\bar{\mathcal{A}}_1^g := \sum_{n=1}^{2} \frac{1}{2\,n} \Big[(C\mathcal{A}_{0n} \cdot \nabla_{\mathfrak{z}} - Sa_{1n}) \Big(\frac{S\mathcal{A}_{0n}}{a_0} \Big) - (S\mathcal{A}_{0n} \cdot \nabla_{\mathfrak{z}} + Ca_{1n}) \Big(\frac{C\mathcal{A}_{0n}}{a_0} \Big) \Big] . \tag{4.16}$$

Lemma 4.1. *[computation of the source term $\bar{\mathrm{A}}_1$] We have:*

$$\partial_t \mathrm{s}(z_0; t(z_0; s))\ d_{\mathfrak{z}} \bar{\Xi}_0(\mathfrak{z}; s)\ \bar{\mathrm{A}}_1(z_0 \,|\, \mathfrak{z}; s) = \bar{\mathcal{A}}_1\big(\bar{\Xi}_0(\mathfrak{z}; s)\big) + \bar{\mathcal{A}}_1^g\big(\bar{\Xi}_0(\mathfrak{z}; s)\big) . \tag{4.17}$$

In particular, with $S\mathcal{A}_{01} = {}^t(S\mathcal{A}_{01}^1, \cdots, S\mathcal{A}_{01}^4)$, we find:

$$\bar{\mathrm{A}}_1^1 = \frac{1}{\partial_t \mathrm{s}}\ \bar{\mathcal{A}}_1^1 \circ \bar{\Xi}_0 - \frac{1}{2\,\partial_t \mathrm{s}}\ (S\mathcal{A}_{01}^2\ \partial_\chi + S\mathcal{A}_{01}^4\ \partial_\varsigma + Ca_{11}) \Big(\frac{C\mathcal{A}_{01}^1}{a_0} \Big) . \tag{4.18}$$

Proof. From (3.93), (3.109) and (3.112), we can infer that:

$$\partial_t \mathrm{s}\ d_{\mathfrak{z}} \bar{\Xi}_0\ \bar{\mathrm{A}}_1 = \overline{\mathcal{A}_1 + (\partial_\upsilon^{-1} \mathcal{A}_0^* \cdot \nabla_{\mathfrak{z}} - a_1)(a_0^{-1}\, \mathcal{A}_0^*)} . \tag{4.19}$$

In view of (3.33a), the function $a_0(\cdot)$ does not depend on υ. The decompositions (4.14) with $j = 0$ and (4.15) with $j = 1$ combined with (4.19) lead directly to (4.17). Moreover, with (2.25), the formula (3.31) gives:

$$\mathcal{A}_0^1 \equiv \mathcal{A}_0^{1*} = \mathrm{w}\ \mathrm{D}_{cf1}^f(F)\ \cos\varsigma\ \cos\upsilon = C\mathcal{A}_{01}^1\ \cos\upsilon\, , \qquad \frac{C\mathcal{A}_{01}^1}{a_0} = \frac{\mathrm{w}\,\mathrm{D}_{cf1}^f}{c(\mathrm{w})\, b}\ \cos\varsigma\, . \tag{4.20}$$

Since $S\mathcal{A}_{01}^1 \equiv 0$, $S\mathcal{A}_{01}^3 \equiv 0$ and $\bar{\Xi}_0^1(\mathfrak{z}) = \mathfrak{z}^1 \equiv \psi$, we recover (4.18). ☐

The possible cancellations induced by the action of the mean operator $\langle\cdot\rangle$ on $\bar{\mathrm{A}}_1(\cdot)$ seem more difficult to evaluate. The effect is to replace the functional $\bar{\mathrm{A}}_1(\cdot)$ by its mean values along the energy curves of $\mathrm{H}(\psi,\cdot)$. The resulting expression $\langle\bar{\mathrm{A}}_1\rangle(\cdot)$ is therefore a function depending only on ψ, w and $\mathrm{H}(\psi,\chi,\varsigma)$, with $\mathrm{H}(\cdot)$ as in (3.44).

The right hand side of (4.17) is the sum of two terms:

1. (1) The first term $\bar{\mathcal{A}}_1$ takes into account the impact of the electromagnetic perturbation. As a consequence of (1.4), (3.28), (3.32b) and (3.32c), we simply find $\bar{\mathcal{A}}_1 \equiv 0$ when $(\mathbf{E},\mathbf{B}) \equiv (0,0)$. The introduction of $(\mathbf{E},\mathbf{B}) \neq (0,0)$ can have a range of effects. Amongst other things, it can induce a displacement of $\langle\bar{\mathfrak{Z}}_0^3\rangle$. Looking at (4.13), this would mean a modification of the kinetic energy of charged particles;
2. (2) The second term $\bar{\mathcal{A}}_1^g$ reflects the influence of the curved geometry related to the inhomogeneities of the exterior magnetic field. Since $\mathcal{A}_0^3 \equiv 0$, we have $\bar{\mathcal{A}}_1^{g3} \equiv 0$. In the absence of $(\mathbf{E},\mathbf{B})$, this implies that $\langle\bar{\mathfrak{Z}}_0^3\rangle(\tau) = \mathrm{w}_0$ for all $\tau \in [0,\mathcal{T}]$. Then, as expected, the kinetic energy is a conserved quantity. Then, the equation (4.10) is concerned only with the evolution of the three components ψ, χ and ς.

4.1.2. *Towards dynamical criteria for the long time confinement.* Simplified models for the poloidal flux function $\psi(\cdot)$ are usually obtained by solving the Grad-Shafranov equation, which is the equilibrium equation in ideal magnetohydrodynamics (MHD). To test the validity of $\psi(\cdot)$, the dynamics of charged particles can also be considered. Trapped and passing particles respond differently. The dynamical criteria of stability depend on Ω_j.

Fix $j \in \{0, \cdots, m\}$, and work with $z_0 \in K^i \subset \Omega_j$. The evolution of $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)$ provides information on the stability of the configuration Ω_j. Of special interest is the behavior of the first component $\langle \bar{\mathfrak{Z}}_0^1 \rangle(\cdot)$ because it controls the spatial confinement properties. Indeed, the portion of the plasma that is issued from $z_0 \in K^i \subset \Omega_j$ remains trapped during long times $\tau \simeq 10^{-4}\, s$ inside the toroidal chamber (of minor radius r_0) if:

$$\langle \bar{\mathfrak{Z}}_0^1 \rangle(z_0; \tau) \leq r_0 \, , \qquad \forall \, (z_0, \tau) \in K^i \times [0, \mathcal{T}] \, . \tag{4.21}$$

The toy model is of course a restrictive case but it is nonetheless instructive.

Example 7. *[toy model] Assume as before that $\psi(r,\theta) = r$ and that $\chi(r,\theta) = \theta$, but also that $(\mathbf{E}, \mathbf{B}) \equiv (0,0)$. Then, by construction:*

$$\langle \bar{\mathrm{A}}_1 \rangle(\psi, \mathrm{w}, \varsigma) = \frac{1}{2\,\pi} \int_0^{2\,\pi} \bar{\mathcal{A}}_1^g(\psi, \chi, \mathrm{w}, \varsigma) \; d\chi \, , \qquad (\psi, \chi) \equiv (r, \theta) \, . \tag{4.22}$$

On the other hand, with Lemma 3.1, noting $\nabla = \partial_r \, e_r + r^{-1} \, \partial_\theta \, e_\theta + R^{-1} \, \partial_\phi \, e_\phi$, we have:

$$S\mathcal{A}_{01}^2 = \mathrm{w} \; r^{-1} \, \cos \omega(r) \; \cos \varsigma \, , \tag{4.23a}$$

$$S\mathcal{A}_{01}^4 = \mathrm{w} \; (e_{f3}^m \cdot \nabla)^t e_{f3}^m \cdot e_{f2}^m \; \sin \varsigma \, , \tag{4.23b}$$

$$Ca_{11} = -\,\mathrm{w} \; (e_{f3}^m \cdot \nabla)^t e_{f3}^m \cdot e_{f2}^m \; \sin^2 \varsigma \; (\cos \varsigma)^{-1} + \mathrm{w} \; (e_{f1}^m \cdot \nabla)^t e_{f2}^m \cdot e_{f1}^m \; \cos \varsigma \, . \tag{4.23c}$$

Remark that $\omega(\cdot)$ depends only on r. Then, with (2.2), it is easy to compute:

$$(e_{f3}^m \cdot \nabla)^t e_{f3}^m \cdot e_{f2}^m = -\,\mathrm{R}^{-1} \, \cos \omega \; \sin \theta \, , \tag{4.24a}$$

$$(e_{f1}^m \cdot \nabla)^t e_{f2}^m \cdot e_{f1}^m = \partial_r (\cos \omega \, e_\theta - \sin \omega \, e_\phi) \cdot e_r = 0 \, . \tag{4.24b}$$

From (3.80), we have $\partial_t \mathrm{s} = \mathrm{w}_0 \; r^{-1} \, \sin \varsigma_0$, and therefore:

$$\langle \bar{\mathrm{A}}_1^1 \rangle(\psi, \mathrm{w}, \varsigma) = -\,\frac{r}{a(r)} \; \cos \omega(r) \; \frac{\mathrm{w}_0}{c(\mathrm{w}_0)} \; \sin \varsigma \; \frac{1}{2\,\pi} \int_0^{2\,\pi} \sin \chi \; d\chi = 0 \, . \tag{4.25}$$

The function $\bar{\mathrm{A}}_1^1(\cdot)$ is not zero. But (4.25) says that its integration with respect to χ gives zero. It follows that $\langle \bar{\mathfrak{Z}}_0^1 \rangle(z_0; \tau) = \psi_0$ for all $\tau \in [0, \mathcal{T}]$. The criterion (4.21) is obviously satisfied. Moreover, the system (4.10) becomes:

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle \; - \; \langle \bar{\mathrm{A}}_1 \rangle \big(\psi_0, \mathrm{w}_0, \langle \bar{\mathfrak{Z}}_0^4 \rangle \big) \; = \; 0 \, , \qquad \langle \bar{\mathfrak{Z}}_0 \rangle(0) = z_0 \; . \tag{4.26}$$

The function $\langle \bar{\mathrm{A}}_1 \rangle(\psi, \mathrm{w}, \cdot)$ is periodic of period $2\,\pi$. If $\langle \bar{\mathrm{A}}_1^4 \rangle(\psi, \mathrm{w}, \cdot)$ has no fixed point:

$$\nexists \, \varsigma \in \mathbb{T} \, ; \qquad \langle \bar{\mathrm{A}}_1^4 \rangle(\psi_0, \mathrm{w}_0, \varsigma) = 0 \, , \tag{4.27}$$

the fourth component $\langle \bar{\mathfrak{Z}}_0^4 \rangle(z_0; \tau)$ is periodic, whereas the second component $\langle \bar{\mathfrak{Z}}_0^2 \rangle(z_0; \tau)$ is linear plus periodic in the sense of Definition 3.2.

The system (4.10) gives access to the instantaneous speed of $\langle \bar{\mathfrak{Z}}_0^1 \rangle$, which is $\langle \bar{\mathrm{A}}_1^1 \rangle(\langle \bar{\mathfrak{Z}}_0 \rangle)$ with the function $\bar{\mathrm{A}}_1^1(\cdot)$ of (4.18). Charged particles tend to expand in the regions of the phase space where $0 < \langle \bar{\mathrm{A}}_1^1 \rangle$, and they are compressed where on the contrary $\langle \bar{\mathrm{A}}_1^1 \rangle < 0$.

Example 7 gives a mathematical confirmation of a well-known empirical law concerning *passing particles* in tokamaks. The rotations around the magnetic axis which are induced by the poloidal field can cancel the local non trivial drift effects. The good and bad curvatures, that is the various signs of $\langle \bar{\mathrm{A}}_1^1 \rangle(\cdot)$, average to produce some stability. Note that a similar property is not expected when dealing with *trapped particles*. The full test of (4.21) would require to solve (analytically or numerically) the system (4.10). Such a general study would be very interesting but it is beyond the scope of the present investigation.

4.1.3. *The WKB method.* In order to find the $\mathfrak{Z}_j(\cdot)$ for $j \geq 1$, we introduce the following inductive hypothesis:

$$(H_j):\ \begin{cases} \textit{The expressions } \mathfrak{Z}_k \textit{ are known on the domain } [0,\mathcal{T}] \times \mathbb{T}^2 \textit{ for } 0 \leq k \leq j-2\,, \\ \textit{The expressions } \bar{\mathfrak{Z}}^\star_{j-1},\ \mathfrak{Z}^*_{j-1} \textit{ and } \mathfrak{Z}^*_j \textit{ are known on the domain } [0,\mathcal{T}] \times \mathbb{T}^2. \end{cases}$$

We have already checked from (4.8), (4.9) and (4.11) the validity of (H_2). Assume (H_k) for all $k \in \{2, \cdots, j\}$. Then, consider what comes in (4.7) with ε^{j-1} in factor. This is a linearized version of (4.9), namely:

$$\partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle + \partial_s \bar{\mathfrak{Z}}^\star_j + (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \mathrm{a}_0\ \partial_v \mathfrak{Z}^*_2 + \mathrm{a}_0\ \partial_v \mathfrak{Z}^*_{j+1} - (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \mathrm{A}_1 = \mathcal{G}_{j-1}\,, \tag{4.28}$$

where $\mathcal{G}_{j-1}$ is a known function. By averaging the constraint (4.28) in $s \in \mathbb{T}$ and $v \in \mathbb{T}$, we recover the linear equation :

$$\partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle - (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \langle \bar{\mathrm{A}}_1 \rangle = \langle \bar{\mathcal{G}}_{j-1} \rangle\,, \tag{4.29}$$

which must be completed by the initial data coming from (4.3):

$$\langle \bar{\mathfrak{Z}}_{j-1} \rangle(0) = \frac{1}{(j-2)!}\ (\partial_\varepsilon^{j-2} \mathfrak{z}_1)(0, z_0; v_0) - \bar{\mathfrak{Z}}^\star_{j-1}(0,0) - \mathfrak{Z}^*_{j-1}(0,0,v_0)\,. \tag{4.30}$$

The *linear* Cauchy problem (4.29)-(4.3) has a solution $\langle \bar{\mathfrak{Z}}_{j-1} \rangle$ on the *whole* interval $[0,\mathcal{T}]$. It means that the life span of all the $\mathfrak{Z}_j$ is the same as the one of (4.10). Stop the induction when $j = N$. With the $\mathfrak{Z}^j$ thus obtained, build $\mathfrak{Z}_\varepsilon$ as indicated in (4.2). Using Borel's summation, one can also construct approximate solutions at order $O(\varepsilon^\infty)$.

4.1.4. *Construction of the profile $\mathcal{V}_\varepsilon$.* Consider the asymptotic expansion proposed in (4.2), and perform a formal analysis at the level of (4.6). The first contribution to appear has the power ε^{-1} in factor. It is:

$$\partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle + \partial_s \bar{\mathcal{V}}^\star_0 + \mathrm{a}_0(\langle \bar{\mathfrak{Z}}_0 \rangle(\tau); s)\ \partial_v \mathcal{V}^*_1 - \mathrm{a}_0(\langle \bar{\mathfrak{Z}}_0 \rangle(\tau); s) = 0\,. \tag{4.31}$$

From (4.31), we can easily extract:

$$\langle \bar{\mathcal{V}}_{-1} \rangle(\tau) = \int_0^\tau \langle \mathrm{a}_0 \rangle(\langle \bar{\mathfrak{Z}}_0 \rangle(r))\ dr = \int_0^\tau \langle (\partial_\tau \mathrm{s})^{-1} \rangle(\langle \bar{\mathfrak{Z}}_0 \rangle(r))\ a_0(\langle \bar{\mathfrak{Z}}_0 \rangle(r))\ dr\,, \tag{4.32}$$

as well as:

$$\bar{\mathcal{V}}^\star_0 = (\partial_s^{-1} \mathrm{a}^\star_0)(\langle \bar{\mathfrak{Z}}_0 \rangle(\tau); s)\,, \qquad \mathcal{V}^*_1 \equiv 0\,. \tag{4.33}$$

Consider the following inductive hypothesis $(\tilde H_j)$ which is closely modelled on (H_j):

$$(\tilde H_j): \begin{cases} \textit{The expressions } \boldsymbol{\mathcal{V}}_k \textit{ are known on the domain } [0,\mathcal{T}]\times\mathbb{T}^2 \textit{ for } -1\le k\le j-2\,, \\ \textit{The expressions } \bar{\boldsymbol{\mathcal{V}}}^\star_{j-1},\ \boldsymbol{\mathcal{V}}^*_{j-1} \textit{ and } \boldsymbol{\mathcal{V}}^*_j \textit{ are known on the domain } [0,\mathcal{T}]\times\mathbb{T}^2. \end{cases}$$

The condition $(\tilde H_1)$ is satisfied. Assume $(\tilde H_k)$ for all $k\in\{1,\cdots,j-1\}$. Then, consider what comes in (4.6) with ε^{j-2} in factor. We find:

$$\partial_\tau\langle\bar{\boldsymbol{\mathcal{V}}}_{j-2}\rangle+\partial_s\bar{\boldsymbol{\mathcal{V}}}^\star_{j-1}+\mathrm{a}_0\,\partial_\upsilon\boldsymbol{\mathcal{V}}_j=\mathcal{K}_{j-2}\,, \tag{4.34}$$

where $\mathcal{K}_{j-2}$ is a known function. The condition (4.34) together with what comes from the constraint $\nu_\varepsilon(0)\equiv\varepsilon\,\upsilon_0$ in (3.101) allows to determine $\langle\bar{\boldsymbol{\mathcal{V}}}_{j-2}\rangle$ through:

$$\partial_\tau\langle\bar{\boldsymbol{\mathcal{V}}}_{j-2}\rangle=\langle\bar{\mathcal{K}}_{j-2}\rangle\,,\qquad\langle\bar{\boldsymbol{\mathcal{V}}}_j\rangle(0)=\upsilon_0\,\delta_{j0}-\bar{\boldsymbol{\mathcal{V}}}^\star_j(0,0)-\boldsymbol{\mathcal{V}}^*_j(0,0,\upsilon_0)\,. \tag{4.35}$$

Then, we can extract:

$$\bar{\boldsymbol{\mathcal{V}}}^\star_{j-1}=\partial_s^{-1}\bar{\mathcal{K}}^\star_{j-2}\,,\qquad\boldsymbol{\mathcal{V}}^*_j=\partial_\upsilon^{-1}(\mathrm{a}_0^{-1}\,\mathcal{K}^*_{j-2})\,. \tag{4.36}$$

From (4.35) and (4.36), we can recover $(\tilde H_j)$. We can therefore determine the profiles $\boldsymbol{\mathcal{V}}_j$ successively for $j=1$ up to N. Put the $\boldsymbol{\mathcal{V}}_j$ together to find $\boldsymbol{\mathcal{V}}_\varepsilon$.

4.2. **Exact solutions.** In Paragraph 4.2.1, we compare the approximate solution ${}^t(\mathfrak{z}^a_\varepsilon,\nu^a_\varepsilon)$ of (4.1) to the exact solution ${}^t(\mathfrak{z}_\varepsilon,\nu_\varepsilon)$ of (3.101). This is Proposition 4.2 stated below. In Paragraph 4.2.2, we describe precisely the oscillating structure of ${}^t(\mathfrak{z}_\varepsilon,\nu_\varepsilon)$. Finally, in Paragraph 4.2.3, we complete the proof of Theorem 1. In fact, this amounts to interpret Proposition 4.2 in terms of the original phase space variables ${}^t(x,v)$.

4.2.1. *Justification of the WKB analysis.* The proof of stability requires only minor changes in comparison to [13]. However, for the sake of completeness, it is reported here. The lifespan of the solution ${}^t(\mathfrak{z}_\varepsilon,\nu_\varepsilon)$ to (3.101) is some $\mathcal{T}_\varepsilon\in\mathbb{R}^*_+$. If $\mathcal{T}<\mathcal{T}_\varepsilon$, then just rename $\mathcal{T}_\varepsilon\equiv\mathcal{T}$. Select some integer $N\in\mathbb{N}^*$, and construct profiles ${}^t(\mathfrak{Z}_\varepsilon,\boldsymbol{\mathcal{V}}_\varepsilon)$ like in (4.2), through the procedure of Subsection 4.1. The substitution of s and υ with $\varepsilon^{-1}\,\tau$ and $\varepsilon^{-1}\,\nu_\varepsilon$ gives access on $[0,\mathcal{T}_\varepsilon]$ to the approximate solution ${}^t(\mathfrak{z}^a_\varepsilon,\nu^a_\varepsilon)$, and then to the remainder ${}^t(r^{\mathfrak{z}}_\varepsilon,r^\nu_\varepsilon)$ of (4.1) which must satisfy:

$$\partial_\tau\begin{pmatrix}r^{\mathfrak{z}}_\varepsilon\\ r^\nu_\varepsilon\end{pmatrix}=\begin{pmatrix}\mathcal{R}^{\mathfrak{z}}\\ \mathcal{R}^\nu\end{pmatrix}\Big(\varepsilon,r^{\mathfrak{z}}_\varepsilon;\tau,\frac{\tau}{\varepsilon},\frac{\nu^a_\varepsilon}{\varepsilon}+\varepsilon^{N-1}\,r^\nu_\varepsilon\Big)\,,\qquad\begin{pmatrix}r^{\mathfrak{z}}_\varepsilon\\ r^\nu_\varepsilon\end{pmatrix}(0)=O(1), \tag{4.37}$$

where:

$$\begin{aligned}(\mathcal{R}^{\mathfrak{z}},\mathcal{R}^\nu)(\varepsilon,r;\tau,s,\upsilon)\,:=\,&\varepsilon^{-N-2}\,\big[\mathrm{a}(\varepsilon,\mathfrak{Z}_\varepsilon;s,\upsilon)-\mathrm{a}(\varepsilon,\mathfrak{Z}_\varepsilon+\varepsilon^N\,r;s,\upsilon)\big]\,\partial_\upsilon(\mathfrak{Z}_\varepsilon,\boldsymbol{\mathcal{V}}_\varepsilon)\\ &+\varepsilon^{-N-1}\,\big[(\mathrm{A},\mathrm{a})(\varepsilon,\mathfrak{Z}_\varepsilon+\varepsilon^N\,r;s,\upsilon)-(\mathrm{A},\mathrm{a})(\varepsilon,\mathfrak{Z}_\varepsilon;s,\upsilon)\big]-(R^{\mathfrak{z}},R^\nu)(\varepsilon;\tau,s,\upsilon)\,.\end{aligned}$$

Proposition 4.2. *[justification of asymptotic expansions associated with the phase ν_ε] For all $\varepsilon\in\,]0,\varepsilon_0]$, the lifespan $\mathcal{T}_\varepsilon$ of the solution to (3.101) is such that $\mathcal{T}_\varepsilon=\mathcal{T}$. Moreover, with a precision in L^∞ that is uniform with respect to $\tau\in[0,\mathcal{T}]$ and initial data z_0 in a compact set K as in (3.75), we can assert that:*

$$\begin{pmatrix}\mathfrak{z}_\varepsilon\\ \varepsilon\,\nu_\varepsilon\end{pmatrix}(\tau)-\begin{pmatrix}\mathfrak{Z}_\varepsilon\\ \varepsilon\,\boldsymbol{\mathcal{V}}_\varepsilon\end{pmatrix}\Big(\tau,\frac{\tau}{\varepsilon},\frac{\nu_\varepsilon(\tau)}{\varepsilon}\Big)=O\Big(\varepsilon^N\Big)\,. \tag{4.38}$$

Proof. Fix $\mathrm{R} \in [1,+\infty[$ large enough. Let $\tilde{\mathcal{T}}_\varepsilon \in [0,\mathcal{T}_\varepsilon]$ be the maximum time such that:

$$|r_\varepsilon^\dagger(0)| \leq \mathrm{R}\,, \qquad |r_\varepsilon^\dagger(\tau)| \leq 2\,\mathrm{R}\,, \qquad \forall\,\tau \in [0,\tilde{\mathcal{T}}_\varepsilon]\,, \qquad \forall\,\dagger \in \{\mathfrak{z},\nu\}\,. \tag{4.39}$$

Recall that $\partial_\upsilon \mathfrak{Z}_\varepsilon = O(\varepsilon^2)$ and that $\mathrm{A}_0 \equiv 0$. Thus, there is a constant $C \in \mathbb{R}_+^*$ such that:

$$\sup\, \{\, |\mathcal{R}^{\mathfrak{z}}(\varepsilon,r;\tau,s,\upsilon)|\,;\, (\varepsilon,r,\tau,s,\upsilon) \in]0,\varepsilon_0] \times B(0,2\,\mathrm{R}) \times [0,\mathcal{T}] \times \mathbb{T}^2\,\} \leq C\,\,(|r|+1)\,.$$

By applying Gronwall's lemma to the first line of (4.37), we find easily:

$$\sup\, \{|r_\varepsilon^{\mathfrak{z}}(\tau)|\,;\, (\varepsilon,\tau) \in]0,\varepsilon_0]\times]0,\tilde{\mathcal{T}}_\varepsilon]\} \,\leq\, (\mathrm{R}+1)\,\, e^{C\,\mathcal{T}} \,<\, +\infty\,. \tag{4.40}$$

Since $\mathrm{a}_0 \not\equiv 0$, the control on $\mathcal{R}^\nu$ is not as good as for $\mathcal{R}^{\mathfrak{z}}$. We can only assert that:

$$\sup\, \{\, |\varepsilon\,\, \mathcal{R}^\nu(\varepsilon,r;\tau,s,\upsilon)|\,;\, (\varepsilon,r,\tau,s,\upsilon) \in]0,\varepsilon_0] \times B(0,2\,\mathrm{R}) \times [0,\mathcal{T}] \times \mathbb{T}^2\,\} \leq C\,\,(|r|+1)\,.$$

The right hand term of (4.37) depends on the component r_ε^ν through the fast variable υ, whereas uniform estimates in $\upsilon \in \mathbb{T}$ are available. This prevents to be faced with a finite time blow-up of $r_\varepsilon^\nu(\cdot)$ before $\tilde{\mathcal{T}}_\varepsilon$. As a matter of fact:

$$\sup\, \{\, |\varepsilon\, r_\varepsilon^\nu(\tau)|\,;\, (\varepsilon,\tau) \in]0,\varepsilon_0] \times [0,\tilde{\mathcal{T}}_\varepsilon]\} \,\leq\, \mathrm{R} + C\,\,(2\,\mathrm{R}+1)\,\,\mathcal{T} \,<\, +\infty\,. \tag{4.41}$$

If necessary, restrict $\mathcal{T} \in \mathbb{R}_+^*$ in order to have:

$$\max\,\big(\,(\mathrm{R}+1)\,e^{C\,\mathcal{T}}\,;\, \mathrm{R} + C\,\,(2\,\mathrm{R}+1)\,\,\mathcal{T}\,\,\big) < 2\,\mathrm{R}\,. \tag{4.42}$$

By this way, we get a contradiction about the definitions of $\tilde{\mathcal{T}}_\varepsilon$ or $\mathcal{T}_\varepsilon$, except if $\tilde{\mathcal{T}}_\varepsilon \equiv \mathcal{T}_\varepsilon \equiv \mathcal{T}$ for all $\varepsilon \in]0,\varepsilon_0]$. Then, coming back to (4.40) and (4.41), we see that $r_\varepsilon^{\mathfrak{z}}$ and r_ε^ν are indeed remainders, with a loss of precision by a fixed factor ε^{-1} concerning r_ε^ν. □

The sup norm estimate (4.38) compares the exact solution ${}^t(\mathfrak{z}_\varepsilon,\nu_\varepsilon)$ to the profile ${}^t(\mathfrak{Z}_\varepsilon,\boldsymbol{\mathcal{V}}_\varepsilon)$ evaluated at the *exact* phase ν_ε, that is with $\upsilon = \nu_\varepsilon$ but not with $\upsilon = \nu_\varepsilon^a$. The above proof involves no linearization with respect to r_ε^ν. It furnishes some uniform bound on the solution ${}^t(r_\varepsilon^{\mathfrak{z}},r_\varepsilon^\nu)$ to the *nonlinear* system (4.37). However, it says nothing about the linearized equations issued from (4.37), which in this case are strongly unstable.

At this stage, the access to ν_ε is achieved through the implicit relation which implies ν_ε at the level of (4.38). This is not very informative. The exact oscillating content of $\nu_\varepsilon(\cdot)$ remains to be clarified. It would be better to work with a well identified phase. In view of (4.2), a natural candidate is:

$$\psi_\varepsilon^\nu(\tau) := \frac{\langle\bar{\boldsymbol{\mathcal{V}}}_{-1}\rangle(\tau)}{\varepsilon} + \bar{\boldsymbol{\mathcal{V}}}_0(\tau,\frac{\tau}{\varepsilon})\,. \tag{4.43}$$

Lemma 4.2. *[description of the phase ν_ε through an expansion involving a frozen phase] There exist profiles $\boldsymbol{\mathrm{V}}_j(x,v;\tau,s,\upsilon) \in \mathcal{C}^\infty(K\times[0,\mathcal{T}]\times\mathbb{T}^2;\mathbb{R})$ such that:*

$$\nu_\varepsilon(\tau) = \psi_\varepsilon^\nu(\tau) + \sum_{j=1}^{N-2} \varepsilon^j\,\,\boldsymbol{\mathrm{V}}_j\Big(\tau,\frac{\tau}{\varepsilon},\frac{\psi_\varepsilon^\nu(\tau)}{\varepsilon}\Big) + O(\varepsilon^{N-1})\,, \qquad \boldsymbol{\mathrm{V}}_1 \equiv \bar{\boldsymbol{\mathcal{V}}}_1\,. \tag{4.44}$$

Proof. The function $\nu_\varepsilon(\cdot)$ can always be put in the form:

$$\nu_\varepsilon(\tau) = \psi_\varepsilon^\nu(\tau) + \varepsilon\,\,\boldsymbol{\mathrm{V}}_\varepsilon^e\Big(\tau,\frac{\tau}{\varepsilon},\frac{\psi_\varepsilon^\nu(\tau)}{\varepsilon}\Big)\,. \tag{4.45}$$

Then, in view of (4.33), the relation (4.1) on ν_ε is equivalent to:

$$\Big[\boldsymbol{\mathcal{V}}^e_\varepsilon - \bar{\mathcal{V}}_1 - \sum_{j=1}^{N-1} \varepsilon^j \; \mathcal{V}_{j+1}(\upsilon + \boldsymbol{\mathcal{V}}^e_\varepsilon)\Big]_{\mid \upsilon = \frac{\psi^\nu_\varepsilon(\tau)}{\varepsilon}} = \varepsilon^{N-1} \; r^\nu_\varepsilon \,. \tag{4.46}$$

First, consider the condition (4.46) in the absence of the remainder, that is:

$$\mathcal{I}(\varepsilon,\tau,s,\upsilon,\boldsymbol{\mathcal{V}}^a_\varepsilon) := \boldsymbol{\mathcal{V}}^a_\varepsilon - \bar{\mathcal{V}}_1(\tau,s) - \sum_{j=1}^{N-1} \varepsilon^j \; \mathcal{V}_{j+1}(\tau,s,\upsilon + \boldsymbol{\mathcal{V}}^a_\varepsilon) = 0 \,. \tag{4.47}$$

Since $\partial_{\mathcal{V}^a_\varepsilon} \mathcal{I} = 1 + O(\varepsilon)$, the Implicit function Theorem applied to (4.47) determines uniquely the function $\boldsymbol{\mathcal{V}}^a_\varepsilon(\tau,s,\upsilon)$. It says that $\boldsymbol{\mathcal{V}}^a_\varepsilon(\cdot)$ is smooth in $(\varepsilon,\tau,s,\upsilon) \in [0,1] \times [0,\mathcal{T}] \times \mathbb{T}^2$. It can be further expressed as a power series with respect to ε, namely:

$$\boldsymbol{\mathcal{V}}^a_\varepsilon(\tau,s,\upsilon) = \sum_{j=1}^{N-2} \varepsilon^{j-1} \; \boldsymbol{\mathcal{V}}_j(\tau,s,\upsilon) + O(\varepsilon^{N-1}) \,, \qquad \boldsymbol{\mathcal{V}}_1 \equiv \bar{\mathcal{V}}_1 \,. \tag{4.48}$$

Involved in the context of the relation (4.46) with the estimate (4.41) on r^ν_ε in mind, the Implicit function Theorem also guarantees that $\boldsymbol{\mathcal{V}}^e_\varepsilon - \boldsymbol{\mathcal{V}}^a_\varepsilon = O(\varepsilon^{N-2})$. Looking at (4.45), we get easily (4.44). □

It is worth mentioning that the asymptotic description (4.44) of ν_ε arises from the implicit function Theorem on equations on ν_ε, which are written in the fast variables (s,υ). Most importantly, the argument put forward to obtain a control on ν_ε does not require to consider derivatives with respect to (τ,x,υ), but only with respect to $\boldsymbol{\mathcal{V}}$. The error estimate inside (4.44) does not come from a study of the evolution equation (3.101). As a matter of fact, since $\partial_\upsilon \mathrm{a}_1 \not\equiv 0$, the linearized equations issued from (3.101) imply some amplification factor of size ε^{-1}, which seems difficult to control.

The description (4.38) of $\mathfrak{z}_\varepsilon$ is also not fully satisfactory since it involves the unknown ν_ε. However, with (4.44), this difficulty can easily be overcome.

Corollary 4.1. *[description of $\mathfrak{z}_\varepsilon$ through an expansion involving the frozen phase ψ^ν_ε] There exist profiles $\tilde{\mathfrak{Z}}_j \in \mathcal{C}^\infty([0,\mathcal{T}] \times \mathbb{T}^2;\mathbb{R})$ such that:*

$$\mathfrak{z}_\varepsilon(\tau) = \sum_{j=0}^{N-2} \varepsilon^j \; \tilde{\mathfrak{Z}}_j\Big(\tau, \frac{\tau}{\varepsilon}, \frac{\psi^\nu_\varepsilon(\tau)}{\varepsilon}\Big) + O(\varepsilon^{N-1}) \,, \qquad \tilde{\mathfrak{Z}}_0 \equiv \langle \bar{\mathfrak{Z}}_0 \rangle \,, \qquad \tilde{\mathfrak{Z}}_1 \equiv \bar{\mathfrak{Z}}_1 \,. \tag{4.49}$$

Proof. It suffices to substitute at the level of (4.1) the phase ν_ε with the content of (4.44). The new profiles $\tilde{\mathfrak{Z}}_j$ are easily deduced inductively from the $(\mathfrak{Z}_j, \boldsymbol{\mathcal{V}}_j)$ through the identity:

$$\sum_{j=0}^{N-2} \varepsilon^j \; \mathfrak{Z}_j\Big(\tau,s,\upsilon + \sum_{j=1}^{N-2} \varepsilon^{j-1} \boldsymbol{\mathcal{V}}_j(\tau,s,\upsilon)\Big) = \sum_{j=0}^{N-2} \varepsilon^j \; \tilde{\mathfrak{Z}}_j(\tau,s,\upsilon) + O(\varepsilon^{N-1}) \,. \tag{4.50}$$

Recall (4.8). Since $\partial_\upsilon \mathfrak{Z}_0 \equiv 0$ and $\partial_\upsilon \mathfrak{Z}_1 \equiv 0$, the terms with ε^0 and ε^1 in factor yield respectively $\tilde{\mathfrak{Z}}_0 \equiv \langle \bar{\mathfrak{Z}}_0 \rangle$ and $\tilde{\mathfrak{Z}}_1 \equiv \bar{\mathfrak{Z}}_1$. For $j \geq 2$, the expression $\tilde{\mathfrak{Z}}_j$ is a more complicated function of the $\mathfrak{Z}_k$ and $\boldsymbol{\mathcal{V}}_k$ with $k \leq j$. □

4.2.2. *The oscillating structure of the flow.* Theorem 1 helps understand how turbulence works in axisymmetric magnetically confined plasmas. It provides a wealth of information, and it has multiple implications. One possible application is the development of simulation methods. It would be interesting to derive from our asymptotic analysis efficient numerical schemes allowing to capture the oscillating framework. Another line of research has been identified in [13, 14].

The two articles [13] and [14] deal with space plasmas, inside magnetospheres. They contain a version of Theorem 1 in a case where $\mathbf{B}(\cdot)$ is the dipole model of earth's magnetic field. In this different and simpler framework, it has been shown that the geometrical structures of the phases govern (time-space) mesoscopic caustics effects which are at the origin of the intermittent production of electromagnetic waves (sferics, whistlers and hisses), and beyond at the source of some non-diffusive transport [1, 16]. Similar phenomena should appear in the present axisymmetric context. If so, they would be based on a detailed analysis of the oscillating structures underlying (1.6). With this perspective in mind, we now examine carefully the content of (1.6).

Scientific observations on fusion devices are carried out in the time variables t or τ, not with s or $\boldsymbol{\tau}$. They imply the physical phase space variables (x,v) or (F,w), but certainly not $\mathfrak{z}$ or ν. It is therefore necessary to interpret the ansatz (4.2) in terms of (τ,x,v). As explained before, the discussion depends heavily on the initial position (x,v) in the phase space. Thus, we have to distinguish between the Ω_j, for $j\in\{0,\cdots,m\}$.

Working with (x,v) becomes all the more important when the perturbation $(\mathbf{E},\mathbf{B})(\cdot)$ is viewed as a self-consistent electromagnetic field. Indeed, this means to consider the full Vlasov-Maxwell system. Now, the Maxwell part is expressed in macroscopic time-space coordinates, like (τ,x). There is no possibility to perform at its level a change of variables depending really on v, like in the lifting procedure. The passage from $(\boldsymbol{\tau},\mathfrak{z},\nu)$ to (τ,x,v) reveals various types of oscillations. As a matter of fact, the description of ${}^t(x_\varepsilon,v_\varepsilon)(x_0,v_0;\tau)$ involves a multiscale and multiphase representation. It furnishes a deterministic access to the turbulent behavior of the flow. Let us start by studying what is most apparent.

$\diamond$ <u>Large</u> <u>amplitude</u> <u>oscillations</u> inside z_ε at frequencies of the order ε^{-1}. They are related to the periodic behavior in s of the profiles. Since $\partial_s\bar{\mathfrak{Z}}_0\equiv 0$, they are not apparent at leading order when dealing with (4.2). However, they are implemented when coming back to the phase space variable z and also (x,v), as revealed by (4.12). To pass from $\boldsymbol{\tau}$ to τ, we can exploit (3.88) in order to obtain:

$$\boldsymbol{\tau}=\varepsilon\ \mathrm{s}\Big(z_0;\frac{\tau}{\varepsilon}\Big)=\frac{2\,\pi\,\tau}{P(z_0)}+\varepsilon\ \mathrm{s}^\star\Big(z_0;\frac{2\,\pi\,\tau}{\varepsilon\,P(z_0)}\Big)=\frac{2\,\pi\,\tau}{P(z_0)}+O(\varepsilon)\,. \tag{4.51}$$

Recall that $z_0={}^t(\psi_0,\chi_0,\mathrm{w}_0,\varsigma_0)$ with ${}^t(\psi_0,\chi_0)$ as in (3.20a) and with ${}^t(\mathrm{w}_0,\varsigma_0)$ as in (3.21). In view of these relations, the position z_0 can be viewed as a function $z(x_0,v_0)$ of (x_0,v_0).

Definition 4.1. *[the phase $\psi_l(\cdot)$] With $P(\cdot)$ as in (3.83), the phase $\psi_l(\cdot)$ is the function:*

$$\begin{array}{rcl}\psi_l\ :\ [0,\mathcal{T}]\times\Omega_j & \longrightarrow & \mathbb{R}\\ (\tau,x,v) & \longmapsto & \psi_l(\tau,x,v):=2\,\pi\,\tau/\mathtt{P}(x,v)\,,\qquad \mathtt{P}:=P\circ z\,.\end{array} \tag{4.52}$$

Of course, there are as many phases $\psi_l(\cdot)$ as there are indices $j \in \{0, \cdots, m\}$. The link (4.51) between $\mathtt{\tau}$ and τ introduces a smooth dependence on the initial data (x_0, v_0). It makes the phase $\psi_l(\cdot)$ appear. Indeed, with (4.12) and (4.51), we get:

$$z_\varepsilon(x_0, v_0; \tau) = \bar{\Xi}_0\Big(\langle \bar{\mathfrak{Z}}_0\rangle\big(z_0(x_0, v_0); \psi_l\big); \frac{\psi_l}{\varepsilon} + \mathrm{s}^\star\big(z_0(x_0, v_0); \frac{\psi_l}{\varepsilon}\big)\Big) + O(\varepsilon) \ . \tag{4.53}$$

During intermediate times ($t \lesssim 1$), the influence of $\psi_l(\cdot)$ as an oscillation disappears. It is not observable as such as long as $t \lesssim 1$. The reason is that $\psi_l(\cdot)$ is linear in τ so that:

$$\frac{\psi_l(\varepsilon\, t, x, v)}{\varepsilon} = \frac{2\,\pi\, t}{\mathtt{P}(x, v)} \, , \qquad \tau = \varepsilon\, t \ . \tag{4.54}$$

On the contrary, in view of (4.53), during long times ($\tau \simeq 1$), the propagation of the oscillating singularities is *at leading order* completely driven by $\psi_l(\tau, x, v)$. The content of $\psi_l(\cdot)$ is determined by $\mathbf{B}(\cdot)$. It has a clear physical meaning. As indicated in Figure 4, it represents the large amplitude oscillations between the turning points (case $j \leq m-1$ of libration) or around the magnetic axis (case $j = m$ of rotation). ◇

Let us now examine what occurs in terms of oscillations for contributions of size $O(\varepsilon)$. In this respect, the situation is far more complex.

◇ <u>Small</u> <u>amplitude</u> <u>oscillations</u> inside z_ε at frequencies of the order ε^{-2}. They are associated with the introduction of ν_ε. They represent the cumulative and interconnected effects during long times ($\tau \simeq 1$) of the gyrations ($\tau \simeq \varepsilon^2$) and of the librations or rotations ($\tau \simeq \varepsilon$). Look at (4.45), and then at (4.44) and (4.49). These identities indicate that it is no more necessary to resort to $\nu_\varepsilon(\mathtt{\tau})$. Indeed, with (4.43), it suffices to consider:

$$\frac{\nu_\varepsilon(\mathtt{\tau})}{\varepsilon} = \frac{\varepsilon\, \nu_\varepsilon(\mathtt{\tau})}{\varepsilon^2} = \frac{\varepsilon\, \psi_\varepsilon^\nu(\mathtt{\tau})}{\varepsilon^2} + O(1) = \frac{\langle \bar{\mathcal{V}}_{-1}\rangle(\mathtt{\tau}) + \varepsilon\, \bar{\mathcal{V}}_0(\mathtt{\tau}, \frac{\mathtt{\tau}}{\varepsilon})}{\varepsilon^2} + O(1)\, . \tag{4.55}$$

Again, the variable $\mathtt{\tau}$ may be replaced as indicated in (4.51). This reveals new phases.

Definition 4.2. *[the phase $\psi_s^0(\cdot)$] The phase $\psi_s^0(\cdot)$ is the function:*

$$\begin{array}{rcl} \psi_s^0 \,:\, [0, \mathcal{T}] \times \Omega_j & \longrightarrow & \mathbb{R} \\ (\tau, x, v) & \longmapsto & \psi_s^0(\tau, x, v) := \langle \bar{\mathcal{V}}_{-1}\rangle \circ \psi_l(\tau, x, v)\, . \end{array} \tag{4.56}$$

Note that a dependence of $\psi_s^0(\cdot)$ on (z_0, v_0) or on (x_0, v_0) is also introduced through the (non indicated) dependence of $\langle \bar{\mathcal{V}}_{-1}\rangle(\cdot)$ on (z_0, v_0) when solving (4.32). The same applies to the function $\Psi_s^1(\cdot)$ below.

Definition 4.3. *[the profile phase $\Psi_s^1(\cdot)$] The profile phase $\Psi_s^1(\cdot)$ is the function:*

$$\begin{array}{rcl} \Psi_s^1 \,:\, [0, \mathcal{T}] \times \Omega_j \times \mathbb{T} & \longrightarrow & \mathbb{R} \\ (\tau, x, v, s) & \longmapsto & \Psi_s^1(\tau, x, v, s)\, , \end{array} \tag{4.57}$$

where we have introduced the profile:

$$\begin{aligned} \Psi_s^1(\tau, x, v, s) := \ & \partial_{\mathtt{\tau}}\big[\langle \bar{\mathcal{V}}_{-1}\rangle\big]\big(\psi_l(\tau, x, v)\big)\ \mathrm{s}^\star\big(z_0(x, v); s\big) \\ & + \bar{\mathcal{V}}_0\Big(\psi_l(\tau, x, v), s + \mathrm{s}^\star\big(z_0(x, v); s\big)\Big)\, . \end{aligned} \tag{4.58}$$

The *complete frozen phase* $\psi_{s\varepsilon}(\cdot)$ appears after substitution at the level of (4.55) of τ with formula (4.51), and after elimination from (4.55) of the induced $O(1)$-contributions. It is associated with frequencies of size ε^{-2}. It belongs to the weakly nonlinear regime:

$$\psi_{s\varepsilon}(\tau,x,v) := \psi_s^0(\tau,x,v) + \varepsilon\ \Psi_s^1\big(\tau,x,v,\psi_l(\tau,x,v)/\varepsilon\big) = \varepsilon\ \psi_\varepsilon^\nu(\tau) + O(\varepsilon^2)\,. \tag{4.59}$$

It is interesting to consider what remains of $\psi_{s\varepsilon}(\cdot)$ when $\tau \simeq \varepsilon\, t$ with $t \lesssim 1$.

Lemma 4.3. *[the gyrophase] During intermediate times, when $t \simeq 1$, the small amplitude oscillations related to $\psi_{s\varepsilon}(\cdot)$ persist. More precisely, with $z(\cdot)$ viewed as a function of (x,v), and with $\mathrm{Z}(\cdot)$ as in (3.34), it involves a phase $\psi_g(\cdot)$ given by:*

$$\frac{\psi_{s\varepsilon}(\varepsilon\, t,x,v)}{\varepsilon^2} = \frac{\psi_g(t,x,v)}{\varepsilon} + O(1)\,, \qquad \psi_g(t,x,v) := c(\mathrm{w}) \int_0^t b\circ \mathrm{Z}(z;r)\ dr\,. \tag{4.60}$$

Proof. By construction, we have:

$$\psi_{s\varepsilon}(\tau,x,v) = \Big[\langle \bar{\mathcal{V}}_{-1}\rangle(\tau) + \varepsilon\, \bar{\mathcal{V}}_0\big(\tau,\frac{\tau}{\varepsilon}\big)\Big]_{\,|\,\tau=\frac{2\,\pi\,\tau}{\mathrm{P}(x,v)}+\varepsilon\,\mathrm{s}^\star\big(z_0;\frac{2\,\pi\,\tau}{\mathrm{P}(x,v)}\big)} + O(\varepsilon^2)\,. \tag{4.61}$$

Using (4.45), coming back to the time variable t, and exploiting (3.29), we can infer that:

$$\varepsilon^{-1}\,\psi_{s\varepsilon}(\varepsilon\, t,x,v) = \nu_\varepsilon(t) + O(\varepsilon) = \int_0^t a_0\circ \mathrm{Z}\big(z(x,v);r\big)\ dr + O(\varepsilon)\,. \tag{4.62}$$

It suffices to look at (3.33a) to conclude. Clearly, the phase $\psi_g(\cdot)$ comes from the integration along the magnetic field lines of the amplitude $b(\cdot)$ of $\mathbf{B}(\cdot)$. $\square$

The function $\psi_g(\cdot)$ is linear plus periodic in the sense of Definition 3.2, because:

$$\psi_g(t,x,v) = \langle\psi_g\rangle(x,v)\ t + \psi_g^\star(t,x,v)\,, \qquad 0 < \langle\psi_g\rangle(x,v) = \langle b\circ \mathrm{Z}\big(z(x,v);\cdot\big)\rangle\,. \tag{4.63}$$

Example 8. *[toy model] A dependence on $\chi\equiv\theta$ of $b(\cdot)$ remains at the level of (2.14a), through the function $\mathrm{R}(r,\theta) = R_0 + r\cos\theta$ of Paragraph 2.1. It follows that $\psi_g^\star(\cdot)\not\equiv 0$. This property, together with a decomposition similar to (4.63), plays a crucial role in* [13] *in order to exhibit electromagnetic intermittency phenomena.*

The expression $z_\varepsilon(\cdot)$ is entirely determined by (3.99) and (3.100). Using (4.49), (4.44) and (4.51) to replace respectively $\mathfrak{z}_\varepsilon$, ν_ε and τ, we can infer that:

$$z_\varepsilon(\tau,x,v) = Z_\varepsilon\Big(\tau,\frac{\psi_l(\tau,x,v)}{\varepsilon},\frac{\psi_{s\varepsilon}(\tau,x,v)}{\varepsilon^2}\Big) + O\big(\varepsilon^{N-1}\big)\,, \qquad Z_\varepsilon = \sum_{j=0}^{N-2}\varepsilon^j\ Z_j\,, \tag{4.64}$$

where the profiles $Z_j(x,v;\tau,s,\upsilon)\equiv Z_j(\tau,s,\upsilon)$ can be identified inductively through:

$$\begin{aligned}\sum_{j=0}^{N-2}\varepsilon^j\ Z_j = {} & \bar{\Xi}_0\Big(\sum_{j=0}^{N-2}\varepsilon^j\ \tilde{\mathfrak{Z}}_j; s+\mathrm{s}^\star\Big)\\ & + \varepsilon\ \Xi_1^*\Big(\sum_{j=0}^{N-2}\varepsilon^j\ \tilde{\mathfrak{Z}}_j; s+\mathrm{s}^\star, \upsilon + \sum_{j=1}^{N-2}\varepsilon^{j-1}\,\boldsymbol{\nu}_j\Big)\,.\end{aligned} \tag{4.65}$$

In (4.65), it must be understood that $\mathrm{s}^\star \equiv \mathrm{s}^\star\big(z(x,v);s\big)$ and that:

$$\tilde{\mathfrak{Z}}_j \equiv \tilde{\mathfrak{Z}}_j(\psi_l + \varepsilon\, \mathrm{s}^\star, s + \mathrm{s}^\star, v)\,, \qquad \boldsymbol{\mathcal{V}}_j \equiv \boldsymbol{\mathcal{V}}_j(\psi_l + \varepsilon\, \mathrm{s}^\star, s + \mathrm{s}^\star, v)\,. \tag{4.66}$$

In particular, with (4.8), (4.44) and (4.49), we find:

$$Z_0 = \bar{\Xi}_0\big(\langle \bar{\mathfrak{Z}}_0\rangle(\psi_l); s + \mathrm{s}^\star\big)\,, \qquad \partial_v Z_1^* = \partial_v \Xi_1^*\big(\langle \bar{\mathfrak{Z}}_0\rangle(\psi_l); s + \mathrm{s}^\star, v + \bar{\boldsymbol{\mathcal{V}}}_1\big)\,. \tag{4.67}$$

In view of (4.3) and (4.8), we have $\partial_v \mathfrak{Z}_\varepsilon = O(\varepsilon^2)$. On the other hand, after applying the transformation $\Xi(\cdot)$, we find $\partial_v Z_0^* \equiv 0$ and $\partial_v Z_1^* \not\equiv 0$. This implies $\partial_v Z_\varepsilon = O(\varepsilon)$. At the level of z_ε, the oscillations with respect to the phase $\psi_{s\varepsilon}$ remain of small amplitude. $\diamond$

$\diamond$ Return to the original phase space variables ${}^t(x_\varepsilon, v_\varepsilon)$. So far, the component ϕ_ε (toroidal angle) has been set apart. From (3.84), it is easy to deduce that:

$$\phi_\varepsilon(\tau,x,v) = \sum_{j=0}^{N-2} \varepsilon^j\, \Phi_j\Big(\tau, \frac{\psi_l(\tau,x,v)}{\varepsilon}, \frac{\psi_{s\varepsilon}(\tau,x,v)}{\varepsilon^2}\Big) + O(\varepsilon^{N-1})\,, \qquad \partial_v \Phi_0 \equiv 0\,. \tag{4.68}$$

Since $x_\varepsilon = \Sigma_f^c(\psi_\varepsilon, \chi_\varepsilon, \phi_\varepsilon)$, the asymptotic expansions (4.64) and (4.68) lead directly to the spatial part x_ε of (1.6), with as announced $\partial_v X_0 \equiv 0$ and $\partial_s X_0 \not\equiv 0$. As regards the velocity part v_ε, the situation is less favourable. Indeed, large amplitude oscillations are introduced by the filtering method of Paragraph 3.1.4. In any event, we find $\partial_v V_0 \not\equiv 0$ since:

$$v_\varepsilon = \mathrm{w}_\varepsilon \; {}^t O_{cf}^m(\psi_\varepsilon, \chi_\varepsilon, \phi_\varepsilon)\; {}^t\big(\cos\varsigma_\varepsilon\, \cos(\varepsilon^{-1}\,\nu_\varepsilon)\,,\, \cos\varsigma_\varepsilon\, \sin(\varepsilon^{-1}\,\nu_\varepsilon)\,,\, \sin\varsigma_\varepsilon\,\big)\,. \tag{4.69}$$

It should be noticed that, for all $(\varepsilon, x_0, v_0) \in\,]0,1] \times \Omega_j$, the time behavior in τ of ${}^t(x_\varepsilon, v_\varepsilon)$ can be periodic, quasiperiodic, almost periodic or even much more complicated. Not to mention that these properties can vary with ε, x_0 and v_0. The inventory of the oscillations is now exhaustive. Intuitively, the phase ψ_l is related to librations or rotations. As noted in (4.54), it cannot be detected as an oscillation as long as $t \lesssim 1$. On the contrary, the phase $\psi_{s\varepsilon}$ takes into account both the intermediate and long time effects of the gyrations around the field lines. It appears already when $t \simeq 1$. Visually, the presence of ψ_l is probably most easily recognizable since it takes place on the spatial part x_ε with a profile of size $O(1)$. However, from a physical standpoint, both ψ_l and $\psi_{s\varepsilon}$ are pertinent. They both provide the same amount of energy in the following sense. Looking at the derivatives of $z_\varepsilon(\cdot)$ with respect to τ, x or v, they can both give rise to contributions of size $O(\varepsilon^{-1})$. $\diamond$

4.2.3. *Comments on the proof of Theorem 1.* Our main statement 1 is a compilation of the previous results. There was a long work of preparation (Sections 3 and 4) to formulate the problem in terms of $(\mathfrak{z},\nu)$. The asymptotic expansion was justified from the side of $(\mathfrak{z},\nu)$ at the level of Proposition 4.2. We have chosen to work with a fixed precision of the order $O(\varepsilon^N)$. By this way, when we looked at the stability, we were able to keep track of the lost in negative powers of ε. The case $O(\varepsilon^\infty)$ follows from standard arguments [21, 23]. The passage from the formal solution (4.49) to (1.6) has been sketched in Paragraph 4.2.2. It is also important to verify that the estimate (4.38) is not too much destroyed when coming back to $(x_\varepsilon, v_\varepsilon)$. The return map is smooth. Difficulties can only arise at the level of the transformations (3.99) and (4.69) , when replacing $\varepsilon^{-1}\,\nu_\varepsilon$ by $\varepsilon^{-1}\,\nu_\varepsilon^a + \varepsilon^{N-1}\, r_\varepsilon^\nu$. But again, this results in the lost of a fixed power of ε.

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